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The Quasiclassical Langevin Equation and Its Application to the Decay of a Metastable State and to Quantum Fluctuations

U. Eckern,[1] W. Lehr,[1,2] A. Menzel-Dorwarth,[1] F. Pelzer,[1] and A. Schmid[1]

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We report on investigations on the consequences of the quasiclassical Langevin equation. This Langevin equation is an equation of motion of the classical type where, however, the stochastic Langevin force is correlated according to the quantum form of the dissipation-fluctuation theorem such that ultimately its power spectrum increases linearly with frequency. Most extensively, we have studied the decay of a metastable state driven by a stochastic force. For a particular type of potential well (piecewise parabolic), we have derived explicit expressions for the decay rate for an arbitrary power spectrum of the stochastic force. We have found that the quasiclassical Langevin equation leads to decay rates which are physically meaningful only within a very restricted range. We have also studied the influence of quantum fluctuations on a predominantly deterministic motion and we have found that there the predictions of the quasiclassical Langevin equations are correct.

KEY WORDS: Decay of metastable states; colored noise; stochastic equations; path integral.

1. INTRODUCTION

Following Einstein's seminal paper on Brownian motion in 1905, Langevin[(1)] developed in 1908 a detailed model of the dynamics of a Brownian particle. Accordingly, the particle exchanges energy and momentum with its environment in two ways. First, there is a frictional force proportional to the velocity and second, the environment exerts irregular pushes

[1] Institut für Theorie der Kondensierten Materie, Universität Karlsruhe, 7500 Karlsruhe, Federal Republic of Germany.

[2] Permanent address: Robert Bosch GmbH, 7000 Stuttgart, Federal Republic of Germany.

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which tend to sustain thermal motion. Hence, we have (for a single degree of freedom) the Langevin equation

$$m\ddot{x} + m\gamma\dot{x} + V'(x) = \xi(t) \tag{1.1}$$

where $x = x(t)$ is the position of the Brownian particle, m is its mass, and $V(x)$ is an externally supplied potential. The two peculiar features above are the frictional force $-m\gamma\dot{x}$ and the stochastic force $\xi(t)$.

If the environment consists of infinitely many degrees of freedom interacting with the particle, one expects that the stochastic force is Gaussian distributed. In this case, the ensemble averages $\langle\xi(t)\rangle$ and $\langle\xi(t)\,\xi(t')\rangle$ characterize the stochastic process $\xi(t)$, that is, the Langevin force, uniquely. In addition, we assume that the process is stationary; therefore

$$\begin{aligned} \langle\xi(t)\rangle &= 0 \\ \langle\xi(t)\,\xi(t')\rangle &= \tilde{K}(t-t') \end{aligned} \tag{1.2}$$

Concerning the power spectrum

$$\tilde{K}(\omega) = \int dt\, e^{i\omega t}\tilde{K}(t) \tag{1.3}$$

we should observe that in classical physics, the dissipation-fluctuation theorem implies

$$\tilde{K}(\omega) = 2m\gamma kT \qquad (\text{WN}) \tag{1.4a}$$

where T is the temperature of the environment. Following a common terminology, we will call a stochastic source $\xi(t)$ a white noise (WN) force if its power spectrum is independent of frequency.

During the past few decades, the Langevin equation (1.1)—and its related Fokker–Planck equation—have found numerous applications[2] in various fields of natural science. Our interest in Brownian motion has been stimulated mostly by investigations on resistively shunted Josephson junctions, where the difference in the phase of the two superconducting order parameters plays the role of the position $x(t)$ of the Brownian particle. The corresponding Langevin equation has been studied repeatedly in the past[3]; however, it has been argued theoretically[4] and confirmed by experiments[5] that the voltage noise observed at low temperatures has a power spectrum which is not white as implied by Eq. (1.4a), but rather is of the quantum mechanical form (quantum noise)

$$\tilde{K}(\omega) = m\gamma \cdot \hbar\omega \coth\frac{\hbar\omega}{2kT} \tag{1.5}$$

Notice that for small frequencies $\hbar|\omega| \ll kT$, we recover the white noise form (1.4a). In the limit $\hbar|\omega| \gg kT$, however,

$$\tilde{K}(\omega) = m\gamma \cdot \hbar|\omega| \qquad \text{(BN)} \tag{1.4b}$$

and we will then call $\xi(t)$ a blue noise (BN) source.

One should keep in mind that the power spectrum of Eq. (1.5) implies a strong temporal anticorrelation of the Langevin force; in fact, we have

$$\tilde{K}(t) = -\frac{\pi m\gamma}{\hbar}\left(\frac{kT}{\sinh \pi kTt/\hbar}\right)^2 \qquad \text{(BN)} \tag{1.6a}$$

provided that $\omega_c|t| \gg 1$, where ω_c is a high-frequency cutoff, which we have, for the moment, introduced in the noise spectrum. Directly related to this anticorrelation is the fact that

$$\int dt\, \xi(t) = 0 \qquad \text{(BN)} \tag{1.6b}$$

for all realizations.

Calculations aiming to derive the Langevin equation, and particularly its generalization to a quantum particle coupled to a quantum environment, start as a rule from the Hamiltonian of a total system, where, for convenience, the environment may be taken to consist of a set of harmonic oscillators[3] coupled linearly to the particle. It is not difficult, then, to obtain the so-called quantum Langevin equation,[(7)] which is formally similar to Eq. (1.1), where, however, $x(t)$ and $\xi(t)$ are replaced by operators $\hat{x}(t)$ and $\hat{\xi}(t)$. Furthermore, one finds that the ensemble average of the symmetrized form $\langle[\hat{\xi}(t)\,\hat{\xi}(t') + \hat{\xi}(t')\,\hat{\xi}(t)]/2\rangle$ is equal to $\tilde{K}(t-t')$ as defined by Eqs. (1.3) and (1.5). Various efforts toward "quantum noise" and quantum Langevin equations have recently been summarized in ref. 8.

In order to emphasize the difference from the exact quantum theory, Schmid[(6)] has called the c-number Langevin equation (1.1) where the power spectrum of the stochastic force is given by Eq. (1.5) the quasiclassical Langevin equation (QCL). As a characteristic feature of the discussion[4] in ref. 6, note that there the quantum mechanical time evolution of the reduced statistical matrix of the Brownian particle has been written in terms of Feynman path integrals. Corresponding to the two coordinates, say $x \pm \frac{1}{2}y$ of the statistical matrix, two paths[5] $x = x(t)$ and $y = y(t)$ appear

[3] Alternatively, one may attach an infinitely long string to the particle. See, for instance, ref. 6.
[4] We take the opportunity to draw attention to an earlier paper[(9)] where a QCL has also been derived within a path integral approach.
[5] It has also been emphasized in ref. 6 that this two-path feature establishes a close relation to the Keldysh technique[(10)] of quantum field theory. See also ref. 11.

in the functional for the action $-i\mathscr{B}_{\mathrm{QM}}[x(t), y(t)]$. In this path integral representation, the QCL emerges directly as a result of the approximation

$$V(x+\tfrac{1}{2}y)-V(x-\tfrac{1}{2}y)\simeq yV'(x) \tag{1.7}$$

Obviously, the QCL is exact for a Brownian particle with external forces linear in the coordinate. It appears that some aspects—but certainly not all—of quantum systems can be described by the QCL. In refs. 6 and 12, it has been argued that the QCL should give a reasonable account of observable phenomena if the damping is sufficiently large; more precisely, if $\gamma \gg \hbar/mx_B^2$, where x_B is a typical scale for the nonlinearity in the potential $V(x)$. Furthermore, one may expect that the QCL is adequate in cases where quantum noise is a small correction to the predominantly deterministic motion of the Brownian particle; this applies to the situation considered in refs. 4 and 5.

In the following, we wish to study in detail the implications of the QCL, with emphasis on the decay of a metastable state where, initially, the Brownian particle is caught in a potential well. In general, one expects that the probability $P(t)$ to find the particle in the well decays exponentially in time,

$$P(t)=e^{-\Gamma t} \tag{1.8}$$

provided that the decay rate Γ is sufficiently small. Defining the bare oscillation frequency ω_0 and the relaxation rate γ_R of the particle at the bottom of the well, this condition is equal to

$$\begin{gathered}\Gamma \ll \gamma_R, \omega_0 \\ \gamma_R=\min(\gamma, \omega_0^2/\gamma)\end{gathered} \tag{1.9}$$

and we will call such a metastable state a *quasistationary state.* Let us also introduce a standard parametrization of the decay rate

$$\Gamma=\frac{\omega_0}{2\pi} a e^{-b} \tag{1.10}$$

Though the separation in exponential b and prefactor a is not unique, the form (1.10) seems to emerge quite naturally as a result of some asymptotic approximation to be discussed later. In our investigation of decaying states we will consider two types of potentials, both chosen to be antisymmetric, $V(-x)=-V(x)$; see Fig. 1. Specifically, we will consider a cubic potential (cp)

$$\begin{gathered}V(x)=\frac{3}{4} V_B \frac{x}{x_B}\left[1-\frac{1}{3}\left(\frac{x}{x_B}\right)^2\right] \quad \text{(cp)} \\ \omega_0^2=V''(x_A)/m=-V''(x_B)/m=3V_B/2mx_B^2\end{gathered} \tag{1.11}$$

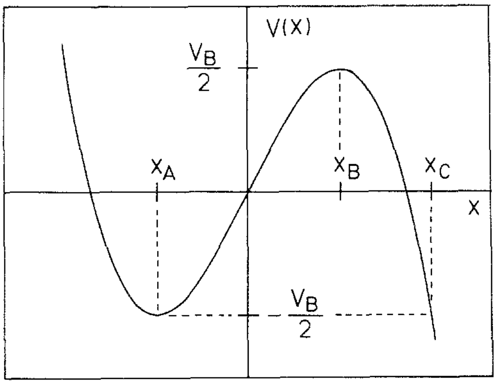

Fig. 1. Potential $V(x)$ featuring a metastable minimum at x_A and a barrier of height V_B at x_B. In quantum tunneling, the exit point x_C plays some role.

and a piecewise parabolic potential (ppp)

$$\begin{aligned} V(x) &= V_B \frac{x}{x_B}\left(1 - \frac{1}{2}\frac{|x|}{x_B}\right) \qquad \text{(ppp)} \\ \omega_0^2 &= V''(x_A)/m = -V''(x_B)/m = V_B/mx_B^2 \end{aligned} \tag{1.12}$$

The contents and the organization of this paper can be summarized as follows. In Section 2, we discuss our numerical simulations.[6] There, realizations of the Langevin force $\xi(t)$ are generated at random in accordance with Eq. (1.2), and the QCL (1.1) is integrated numerically for potentials of the type (1.11) and (1.12). Specifically, a time-discretized QCL is introduced in Section 2.1, and in Section 2.2 we explain how to process the data in order to obtain the decay rate. In Sections 2.3 and 2.4 we present and discuss the decay rates and, to a limited extent, the first passage times for the cubic (cp) and piecewise parabolic potentials (ppp), as a function of the parameters γ/ω_0, V_B/kT, and $kT/\hbar\omega_0$.

In Section 3 we outline the analytical calculations which are based on an asymptotic expansion of path integrals.[7] As a basis, we introduce in Section 3.1 a path integral representation. Of course, we could have immediately started with the representation of ref. 6, but we thought it to be instructive to derive it on the basis of the QCL as introduced above.

[6] Part of these results have already been published in ref. 13.

[7] To our knowledge, such an asymptotic expansion of path integrals for the decay rate was first published by Caroli *et al.*[(14)] for white noise. This asymptotic expansion corresponds, again for white noise, exactly to Kramers' approach of decay[(15)] for moderate to large damping.

Essentially, the QCL defines the mapping $\xi(t) \to x(t)$, and in Section 3.2 we emphasize fundamental properties of the Jacobian $d[\xi(t)]/d[x(t)]$ which follow from the causality principle. Section 3.3 is meant to explain some specific features of path integration as well as the transition from the one-path $[x(t)]$ to the two-path $[x(t), y(t)]$ formalism, which will later allow the transition from the approximate quasiclassical action $\mathscr{B}[x(t), y(t)]$ to the quantum mechanical action[8] $\mathscr{B}_{QM}[x(t), y(t)]$. In Section 3.4, we rewrite the action in terms of reduced variables, whereas Section 3.5 contains the definition of the extremal paths that form the backbone of the asymptotic expansion.

As a rule, the asymptotic expansion is justified for quasistationary states as defined by (1.9). In terms of model parameters, this means essentially that

$$V_B \gg kT_N; \qquad T_N \sim \max(T, T_B) \tag{1.13}$$

where the so-called crossover temperature T_B is, in the limit of strong damping $\gamma \gg \omega_0$,

$$kT_B = \frac{\hbar\omega_0^2}{2\pi\gamma} \tag{1.14}$$

The simplest types of extremal paths are kink and antikink, which are paths that connect the bottom and top of the well. It is a characteristic feature of the Langevin equation that there are interactions between kink and antikink which lead to a combined object. The short-range interaction is analyzed in Section 3.6. Next we discuss in Section 3.7 the Gaussian fluctuations about extremal paths and then we show in Section 3.8 that the fluctuations about the combined object lead to long-ranged attraction ("confinement") between kink and antikink. As a result, we have to handle the integration with respect to the collective coordinate representing the kink and antikink separation very carefully. Eventually, we arrive at the exponential law of decay of the form (1.8) and (1.11), where the prefactor a and exponential b are presented in terms of calculable expressions. For high temperatures $T \gg T_B$, we recover in leading order the white noise (WN) result, where $b = V_B/kT$.

As shown in Section 4, the advantage of the piecewise parabolic potential (ppp) is that all these calculations can be done explicitly. The approach to a self-consistent solution of the extremal paths is outlined in Section 4.1, and in Sections 4.2–4.4 the antikink, the kink, and the combined object are

[8] According to Eq. (5) of ref. 6, the notion is such that $\mathscr{B}_{QM}[x(t), y(t)] = -(i/\hbar)\,\bar{S}^*[\mathbf{x}(t)]$, where $\mathbf{x}(t) = (x(t), y(t))$.

discussed in detail. While the exponential b is directly proportional to the action $\mathscr{B}^k$ of the kink, the prefactor is related to the Gaussian fluctuations. It is also an advantage of the ppp that, as shown in Section 4.5, the fluctuation determinant can be calculated explicitly. At this point, we acknowledge that for the ppp, the decay rate has been calculated by Luciani and Verga[16] by the same methods for a Lorentzian noise, where

$$\tilde{K}^L(\omega) = \frac{2m\gamma kT}{\omega^2\tau^2 + 1} \tag{1.15}$$

Our work is a generalization of the analytical expressions to a noise source of arbitrary color, though in our specific calculations, a quantum noise of the type (1.5) has been assumed. For the sake of completeness, we reproduce the results of ref. 16 for Lorentzian noise in Section 4.6.

Perhaps it is worthwhile to note that at the beginning of our work, we did have some doubts whether a Langevin equation with a noise power increasing indefinitely with frequency, $\tilde{K}(\omega) \propto |\omega|$, has a well-defined meaning. All our investigations suggest that the answer is positive, provided that the acceleration term $m\ddot{x}$ is retained in the equation of motion.

In Section 5, the cubic potential (cp) is considered. Corrections to the white noise limit, which can be computed analytically in the strongly damped limit $\omega_0 \ll \gamma$, are presented in Section 5.1. Although the prefactor can only be calculated for white noise and strong damping, we have outlined such a calculation in Section 5.2, since it has some tutorial value. Numerical calculations for strong damping but arbitrary temperatures are discussed in Section 5.3.

In Section 6, we formulate the problem of calculating the quantum decay in real time by reconsidering our previous calculation, but making use of the quantum mechanical action $\mathscr{B}_{\mathrm{QM}}[x(t), y(t)]$. We find that above the crossover temperature $T > T_B$, an extremal path of the type of a kink exists, but that $\mathscr{B}_{\mathrm{QM}}$ is such that we have always $b = V_B/kT$. Hence, we have to conclude that *even the high-temperature corrections of the exponential b as calculated from the* QCL *are an artefact of the quasiclassical approximation* (1.7).

On the other hand, we show in Section 7 that the QCL reproduces correctly the effects of quantum noise in a Josephson junction as postulated in ref. 4 (and confirmed later experimentally in ref. 5). We summarize our results in the concluding Section 8.

2. NUMERICAL SIMULATIONS OF DECAY

2.1. Quasiclassical Langevin Equation in Discrete Time

The strong temporal correlation of the Langevin force—see Eq. (1.6)—requires that we select a finite (though sufficiently long) time interval $\bar{t} = t_f - t_i$; for convenience, we put the initial time $t_i = 0$ and the final time $t_f = \bar{t}$. Essentially, a simulation comprises three steps: (i) selection of a realization of the Langevin force; (ii) integration of the QCL; and (iii) observation of the time τ required for a decay event.

Since we are only interested in quasistationary states, we will find that the decay time observed in one realization is almost independent of the initial conditions if the initial position and velocity are close to the local equilibrium values of the metastable minimum; in fact, the choice $x_i = x_A$ and $\dot{x}_i = 0$ has proven to be adequate.

We integrate the Langevin equation by solving the discrete-time QCL

$$m(\ddot{x})_n + m\gamma(\dot{x})_n + V'(x_n) = \xi_n \tag{2.1}$$

where n labels the time steps

$$t_n = \Delta t \cdot n, \qquad n = 0, 1, ..., N = \bar{t}/\Delta t \tag{2.2a}$$

and where

$$\begin{aligned} x_n &= x(t_n) \\ (\dot{x})_n &= (x_{n+1} - x_{n-1})/(2\Delta t) \\ (\ddot{x})_n &= (x_{n+1} - 2x_n + x_{n-1})/(\Delta t)^2 \end{aligned} \tag{2.2b}$$

Furthermore, ξ_n is the stochastic force averaged over a time interval Δt:

$$\xi_n = \frac{1}{\Delta t} \int_{-\Delta t/2}^{\Delta t/2} dt' \, \xi(t_n + t') \tag{2.3}$$

We have made extensive tests which show that our results do not depend on the discretization (2.2) or on the temporal average (2.3) of the Langevin force provided that $\Delta t \ll \omega_0^{-1}$.

The best adaption of a correlated noise to a finite time interval is by means of a periodic extension, $\xi(t + \bar{t}) = \xi(t)$. Hence, we put[9]

$$\xi_n = \frac{1}{\bar{t}} \sum_{k=-N/2+1}^{N/2} \xi(\nu_k) \, e^{-i\nu_k t_n}; \qquad \nu_k = \frac{2\pi}{\bar{t}} k \tag{2.4}$$

[9] Strictly speaking, one should define ν_k such that $\nu_{k+N} = \nu_k$. The theory of lattice vibrations suggests that we should choose $\nu_k = 2\,|\sin \pi k/N|/\Delta t = (2/\bar{t})\, N\,|\sin \pi k/N|$. In the present problem, this detail has no influence on the final results.

and choose the complex Fourier amplitudes $\xi(\nu_k)$ to be independent Gaussian random quantities with dispersion

$$\langle |\xi(\nu_k|^2\rangle = \tilde{K}(\nu_k) \tag{2.5}$$

Thus, a realization consists of N quantities $\xi(\nu_k)$ that have been chosen according to the above specification.[10] Eventually, the stochastic force in discrete time is found from Eq. (2.4).

Concerning some technical details, we have typically chosen $\bar{t} \sim 10^4/\omega_0$ and $\Delta t \sim 0.2/\omega_0$ ($N \sim 5 \times 10^4$). Generally, we need to sample only over $\mathcal{N}_0 \sim 10^3$ realizations except in extreme cases (large barrier height or large damping), where it has been necessary to take up to $\mathcal{N}_0 \sim 3 \times 10^4$ realizations for reasonable accuracy.[11]

2.2. The Decay Rate

According to our observations, the simple choice of initial values $x_n = x_A$ for $n \leqslant 0$ is sufficient. Now, for each realization $\xi_n^{(j)}$ of the stochastic force we define a first passage time $\tau^{(j)}(x) = \tau_n^{(j)}(x)$ from the condition $x_{n-1}^{(j)} < x \leqslant x_n^{(j)}$, which is the time the Brownian particle needs to pass the point x. As it turns out, $\tau_n^{(j)}(x)$ is independent of x for $x \gtrsim x_C$, at least on the scale of the decay time Γ^{-1}, provided that $\Gamma \ll \gamma_R$. Therefore, it is justified to identify $\tau^{(j)}(x_C)$ with the decay time of that particular realization.

In the case of a decay exponential in time as given in Eq. (1.8), the inverse Γ^{-1} of the decay rate should be calculated according to

$$\Gamma^{-1} = \frac{1}{\mathcal{N}_0} \sum_{j=1}^{\mathcal{N}_0} \tau^{(j)}(x_C) \tag{2.6a}$$

However, we have to keep in mind that the ensemble (of $\mathcal{N}_0$ realizations) is observed only for a finite time interval $\bar{t}$. Consequently, only $\mathcal{N}_1$ realizations ($\mathcal{N}_1 \leqslant \mathcal{N}_0$) are seen to decay. Therefore, we propose to improve Eq. (2.6a) as follows:

$$\Gamma^{-1} = \frac{1}{\mathcal{N}_1} \left\{ \sum_{j=1}^{\mathcal{N}_1} \tau^{(j)}(x_C) + \bar{t}(\mathcal{N}_0 - \mathcal{N}_1) \right\} \tag{2.6b}$$

The second term is just the mean contribution of events where $\tau(x_C) > \bar{t}$.

[10] We emphasize again that the strongly correlated noise requires the time interval $\bar{t}$ to be fixed at the beginning. There is no need to do so in case of uncorrelated noise (WN) or weakly correlated noise. See, for instance, ref. 16 for Lorentzian noise.

[11] This means that in extreme cases, the Brownian particle is observed altogether for $\sim 10^9$ time steps. The CYBER 205 we are using needs for this job about 1 h of CPU time.

It is convenient to compare the decay rate Γ with the intrinsic frequency $\omega_0/2\pi$ [as already implied in Eq. (1.10)]. Furthermore, on account of its exponential dependence on the parameters, we will use the (negative) logarithmic decay rate

$$\sigma = \ln \frac{\omega_0}{2\pi\Gamma} \tag{2.7a}$$

in the presentation of the numerical data. Note that in terms of the form (1.10)

$$\sigma = b - \ln a \tag{2.7b}$$

2.3. Data for the Cubic Potential (cp)

In Fig. 2, the data (dots) obtained from simulations of the QCL in the white noise limit, i.e., for $\tilde{K}(\omega) = 2m\gamma kT$, are compared with Kramers'[15] moderate damping result (straight lines), where in Eq. (2.7b) we have to insert

$$\begin{aligned} b &= \frac{V_B}{kT} \qquad \text{(WN)} \\ a &= \frac{1}{\omega_0}\left[\left(\omega_0^2 + \frac{1}{4}\gamma^2\right)^{1/2} - \frac{1}{2}\gamma\right] \end{aligned} \tag{2.8}$$

Clearly, our data are in good agreement with the theoretical prediction for $V_B \gtrsim kT$. The fact that for $\gamma = 0.3\omega_0$ and for $V_B \lesssim kT$ the observed value of σ is slightly larger than the theoretical one can be explained by corrections to the prefactor, related to nonequilibrium effects, which are known to be important for $\gamma/\omega_0 \lesssim kT/V_B$.[15,17]

The strong anticorrelation of blue noise—see Eq. (1.6)—manifests itself in a peculiar position dependence of the first passage time which is in contrast to the case of uncorrelated white noise. In Fig. 3, we plot $\sigma(x) = \ln[\langle\tau(x)\rangle\, \omega_0/2\pi]$, where $\langle\tau(x)\rangle$ is the mean value of the first passage time as a function of x, which has been calculated by generalizing Eq. (2.6b). For white noise, $\sigma(x_B)$ is close to its asymptotic value $\sigma(x \gg x_B)$; in fact, the theoretical prediction[12] is $\sigma(x \gg x_B) - \sigma(x_B) = \ln 2$, which means that the particle on the top of the barrier has two equal choices, either going backward or forward. For blue noise, on the other hand, the mean

[12] See, for example, Section 5.2.7 of Gardiner.[2] The prediction is valid only for $\gamma \gg \omega_0$. For $\gamma = 2\omega_0$, we have found that $\sigma(x \gg x_B) - \sigma(x_B) \simeq 0.5 \ln 2$.

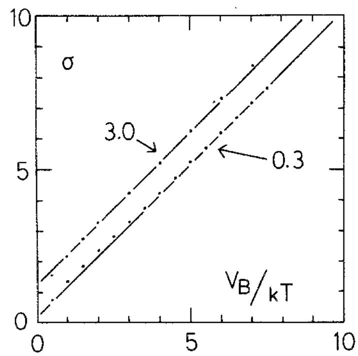

Fig. 2. Numerical data (points) of the logarithmic decay rate $\sigma = \ln(\omega_0/2\pi\Gamma)$ vs. V_B/kT in the white noise limit ($T \gg T_B$) and for the cubic potential (cp). For comparison, we show Kramers' moderate damping result (straight lines) for $\gamma/\omega_0 = 3$ and $\gamma/\omega_0 = 0.3$.

first passage time $\langle \tau(x_B) \rangle$ on the top of the barrier is orders of magnitude smaller than the mean escape time $\langle \tau(x \gtrsim x_C) \rangle$. This difference can be understood as follows. For the particle to reach the top, it requires that the Langevin force is in the positive direction for a comparatively long time; therefore, the temporal anticorrelation of the blue noise implies that subsequently the probability for a negative Langevin force is large, which means that the particle is very likely being pushed back into the well. The results presented in Fig. 3 indicate that the escape has become definite beyond x_C, where the systematic force $-V'(x)$ is large enough to overcome fluctuations in the Langevin force.

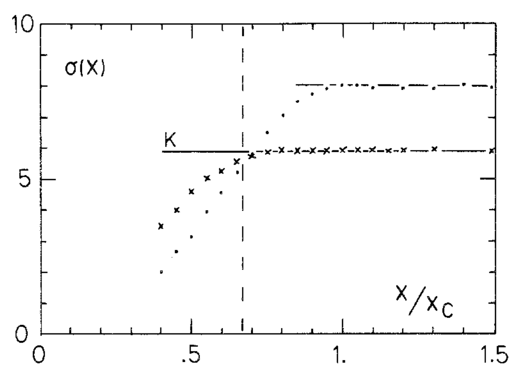

Fig. 3. Logarithmic first passage time $\sigma(x) = \ln(\omega_0 \langle \tau(x) \rangle / 2\pi)$ for white noise ($T = 2.5 T_B$; crosses) and for blue noise ($T = 0$; dots), respectively, in the case of a cubic potential (cp). Other parameters are $\gamma = 2\omega_0$ and $V_B = \hbar\omega_0$. The horizontal line K corresponds to Kramers' result. Dashed vertical line, $x = x_B$.

Each data point in Fig. 3 represents $\mathcal{N}_0 = 10^3$ realizations of the stochastic force. In the blue noise limit and for $x \gtrsim x_C$, we have ~ 600 inconclusive events for $\bar{t} \sim 10^4/\omega_0$. The fluctuations in the data for $x > x_C$ indicate that the statistical error is a few percent when the decay time is small, say $\sigma \gtrsim 7$.

The temperature dependence of the logarithmic decay rate $\sigma = \ln(\omega_0/2\pi\Gamma)$ is shown in Fig. 4 for $V_B = \hbar\omega_0$. We find a rather broad crossover from the white- to the blue-noise-dominated regime roughly at the temperature T_B given in Eq. (1.14). For comparison, the white noise result of Kramers [see Eq. (2.8)] is shown by a solid line.

We have also studied in detail the dependence of the decay rate on the barrier height V_B and on the damping γ in the blue noise limit ($T = 0$). From the data shown in Fig. 5, it is evident that for not too small damping ($\gamma \gtrsim \omega_0$), we have $\sigma \sim (V_B/\hbar\omega_0)(\gamma/\omega_0)$ to a good approximation. For comparison, we show also the result of a calculation of the exponent b (Section 5.3), where in the limit $\gamma/\omega_0 \gg 1$ the result

$$b = 4.2 \frac{\gamma}{\omega_0} \frac{V_B}{\hbar\omega_0} = 0.67 \frac{V_B}{kT_B} \qquad \text{(BN)} \tag{2.9}$$

[cf. Eq. (1.14)] has been obtained. The numerical results for σ are rather close to the expression (2.9). In view of Eq. (1.10), this means that the prefactor $a \sim 1$ (we have not succeeded in calculating the prefactor in this case).

We note that in the same limit Caldeira and Leggett[18] have calculated from a quantum mechanical tunneling theory a decay rate with an exponent $b^{(\mathrm{CL})} = 3\pi(\gamma V_B/\hbar\omega_0^2) = \frac{3}{2} V_B/kT_B$. Thus, we obtain from the

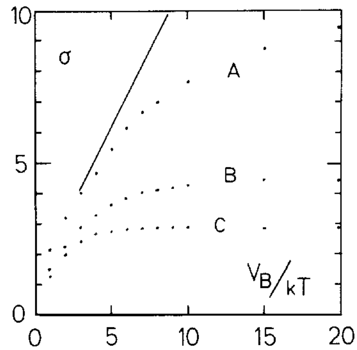

Fig. 4. Temperature dependence of the logarithmic decay $\sigma = \ln(\omega_0/2\pi\Gamma)$ for $V_B = \hbar\omega_0$ and (A) $\gamma = 3\omega_0$, (B) $\gamma = \omega_0$, and (C) $\gamma = 0.3\omega_0$. The numerical data are represented by points; the solid line shows the white noise result of Kramers for $\gamma = 3\omega_0$.

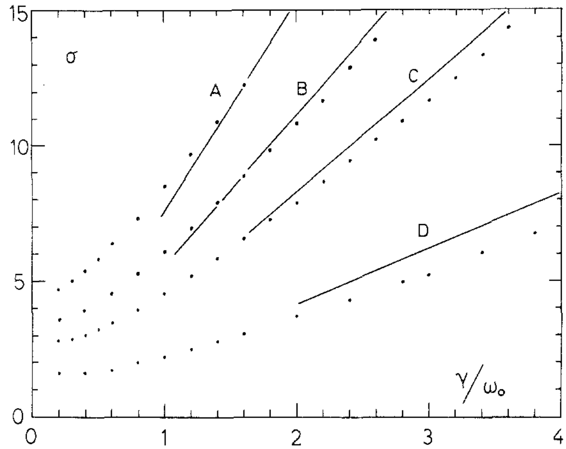

Fig. 5. Numerical data (points) of the logarithmic decay rate $\sigma = \ln(\omega_0/2\pi\Gamma)$ vs. γ/ω_0 for the cubic potential (cp) and for blue noise ($T=0$). The parameters for $V_B/\hbar\omega_0$ are (A) 1.83, (B) 1.33, (C) 1.00, (D) 0.5. The solid lines represent the exponential b of the asymptotic result (2.9).

QCL an exponent of the decay rate which is a factor of $0.67/1.5 \sim 1/2$ smaller than from quantum tunneling.

Blue noise is only meaningful if the mass m is kept finite.[13] Therefore, we understand that the data of a simulation for $m=0$ depend on the discretization. In fact, $(\Delta t)^{-1}$ serves as a high-frequency cutoff which for finite mass is given by $\omega_0 = [V''(x_B)/m]^{1/2}$.

2.4. Data for the Piecewise Parabolic Potential (ppp)

In Fig. 6, the data (dots) obtained from simulations of the QCL in the blue noise limit, i.e., for $\tilde{K}(\omega) = m\gamma\hbar\,|\omega|$, are compared with analytical results (solid lines) of an asymptotic expansion to be presented in Section 4. The agreement is very good if the condition (1.13) for an asymptotic expansion is satisfied. We conclude that the strong increase of the noise power with frequency $\tilde{K}(\omega) \propto |\omega|$ poses no problem in numerical simulation or in analytical calculations provided (see footnote 13) that the acceleration term $m\ddot{x}$ is retained in the Langevin equation.

[13] Compare this with the fact that the width $\langle x^2 \rangle^{1/2}$ of a damped harmonic oscillator diverges logarithmically for $m \to 0$. See also the comment at the end of the paragraph below Eq. (1.15).

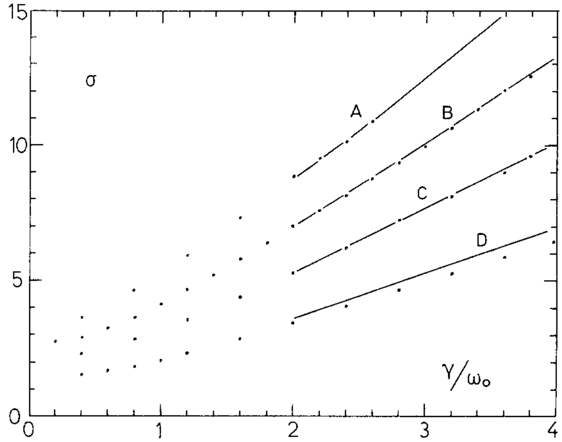

Fig. 6. Numerical data of the logarithmic decay rate $\sigma = \ln(\omega_0/2\pi\Gamma)$ vs. γ/ω_0 for the piecewise parabolic potential (ppp) in the blue noise limit ($T=0$). The parameters for $V_B/\hbar\omega_0$ are (A) 1.25, (B) 1.0, (C) 0.75, (D) 0.5. The solid lines are the result of the asymptotic calculation, which has been carried through for $\gamma/\omega_0 > 2$.

3. ASYMPTOTIC EXPANSION

3.1. Path Integral Representation

Since the Langevin force[14] ξ_t is a Gaussian process, the probability for its realization is a Gaussian functional:

$$W[\xi_t] = \text{const} \cdot \exp -\frac{1}{2} \int dt\, dt'\, \xi_t \tilde{N}_{tt'} \xi_{t'} \tag{3.1}$$

Specifically, $W[\xi_t]\, d[\xi_t]$ is the probability to find a realization in the region $[\xi_t] \cdots [\xi_t + d\xi_t]$ of the function space, and $\tilde{N}_{tt'}$ is the inverse of $\tilde{K}_{tt'}$:

$$\int dt\, \tilde{N}_{t\bar{t}} \tilde{K}_{\bar{t}t'} = \delta(t-t') \tag{3.2}$$

Note that the above ansatz implies the form (1.2) for the correlators. Also, one may convince oneself easily that for a discretization as introduced

[14] In the following, we indicate the time (as well as frequency) dependence of the quantities by a subscript; e.g., $x_t := x(t)$ and $\tilde{K}_{tt'} := \tilde{K}(t-t')$.

in Section 2.1, the functional above is replaced by a multidimensional probability distribution.

At this point we emphasize that we have to consider processes ξ_t in the total time domain $(-\infty, +\infty)$ in order to guarantee stationarity[15] (translational invariance in time). Correspondingly, the time integrations above run along the entire time axis.

We obtain the probability functional $P[x_t]$ for the stochastic process x_t if we interpret[16] the Langevin equation (1.1) as a mapping $\xi_t \to x_t$. *Note that the Jacobian* $d[\xi_t]/[dx_t]$ *of this transformation is a constant independent of* x_t, *provided that* $m \neq 0$. In fact, we will show in Section 3.2 that this property[17] is a consequence of causality: we interpret the Langevin equation (1.1) such that the force ξ_t will influence x_t only at later times. Consequently,

$$P[x_t] = \text{const} \cdot \exp - \mathscr{A}[x_t] \tag{3.3}$$

where the action is given by

$$\begin{aligned} \mathscr{A}[x_t] &= \frac{1}{2} \int dt\, dt'\, \xi[x_t]\, \tilde{N}_{tt'} \xi[x_{t'}] \\ \xi[x_t] &= m\ddot{x}_t + m\gamma\dot{x}_t + V'(x_t) \end{aligned} \tag{3.4}$$

3.2 The Jacobian

Recall that the Jacobian is defined as a determinant

$$\frac{d[\xi_t]}{d[x_t]} = \det \frac{\delta \xi_t}{\delta x_{t'}} \tag{3.5a}$$

where

$$\frac{\delta \xi_t}{\delta x_{t'}} = \delta(t-t')[m\partial_t^2 + m\gamma\partial_t + V''(x_t)] \tag{3.5b}$$

Usually, the δ-function is omitted since its contribution is absorbed in the normalization.

[15] The periodic extension introduced in the numerical simulations of Section 2.1 comes closest to this requirement.

[16] To our knowledge, Graham[(19)] was the first to make use of this interpretation in a systematic way. At this point, we draw attention to a recent paper on path integral formulation of Langevin equations with colored noise by Hänggi.[(20)]

[17] A related feature is found in the Keldysh technique, where the generating functional is always properly normalized (the sum of vacuum diagrams is zero).

Assume that $V''(x_t)$ has a definite limit V''_∞ for $|t| \to \infty$, and that this limit corresponds to some local equilibrium $V''_\infty \geqslant 0$. Correspondingly, we define an unperturbed Green's function

$$(m\partial_t^2 + m\gamma\partial_t + V''_\infty)\, G^0_{tt'} = \delta(t-t') \tag{3.6}$$

If one imposes the condition that $G^0_{tt'}$ remain finite for $|t|, |t'| \to \infty$, one finds that $G^0_{tt'} = 0$ for $t \leqslant t'$. As a simple example, consider the case where $V''_\infty = 0$; then

$$G^0_{tt'} = \theta(t-t')\frac{1}{m\gamma}\,[1 - e^{-\gamma(t-t')}] \tag{3.7}$$

Except for a constant, the Jacobian is equal to

$$\frac{d[\xi_t]}{d[x_t]} = \frac{\det[m\partial_t^2 + m\gamma\partial_t + V''(x_t)]}{\det[m\partial_t^2 + m\gamma\partial_t + V''_\infty]} \tag{3.8}$$

Then, it follows from Eqs. (3.6) and (3.8) that

$$\ln\frac{d[\xi_t]}{d[x_t]} = \mathrm{Sp}\,\ln[\delta(t-t') + G^0_{tt'}\,\Delta V''(x_{t'})] \tag{3.9}$$

where $\Delta V''_t = V''(x_t) - V''_\infty$. In a perturbation expansion, we have

$$\begin{aligned}&\ln[\delta(t-t') + G^0_{tt'}\Delta V''_{t'}]\\ &\quad = G^0_{tt'}\Delta V''_{t'} - \frac{1}{2}\int d\bar{t}\, G^0_{t\bar{t}}\Delta V''_{\bar{t}} G^0_{\bar{t}t'}\Delta V''_{t'} + \cdots\end{aligned} \tag{3.10}$$

Considering the fact that $G^0_{tt'} = 0$ for $t \leqslant t'$, we arrive at the conclusion that

$$\ln\frac{d[\xi_t]}{d[x_t]} = 0 \qquad (m \neq 0) \tag{3.11}$$

Let us now consider the overdamped limit where the acceleration term $m\ddot{x}_t$ is omitted in the Langevin equation. Then, Eq. (3.6) has to be replaced by

$$(m\gamma\partial_t + V''_\infty)\, G^0_{tt'} = \delta(t-t') \tag{3.12a}$$

and one finds that the appropriate solution is discontinuous, since $G^0_{tt'} = 0$ for $t \leqslant t'$, while $G^0_{tt'} = 1/m\gamma$ for $t \to t' + 0$. Let us consider a symmetric formulation (see e.g., Schulman,[21] Chapter 5), where

$$G^0_{tt} = \frac{1}{2m\gamma} \tag{3.12b}$$

Then, we obtain from Eq. (3.10)

$$\ln \frac{d[\xi_t]}{d[x_t]} = \frac{1}{2m\gamma} \int dt\, \Delta V_t''; \qquad m=0 \tag{3.13}$$

which agrees with the form of the Jacobian quoted in the literature(14, 16, 19) for the overdamped (Smoluchowski) limit.[18]

3.3 Path Integration

The probability for a particular event (e.g., decay of a metastable state) to occur is obtained by summing $P[x_t]$ of Eq. (3.3) with respect to all paths x_t that include this event (see Schulman(21) for an introduction to path integrals). Thus

$$P(\text{event}) = \int' d[x_t]\, P[x_t] \tag{3.14a}$$

where the prime on the path integral means the restriction to the appropriate paths.

At this point, we recall the fact that the Langevin force ξ_t is correlated for very distant times. Consquently, the process $(x_t, \dot{x}_t)$ is not Markovian, since the position and the velocity do not specify uniquely an initial state (this point has been made very clear by Hänggi(22). In our approach, this peculiarity shows up mathematically, in the fact that all time integrations run along the entire time axis. Therefore, the path integral has to be done with respect to all paths x_t in the range $(-\infty, +\infty)$ which satisfy the requirements of the event in consideration.[20] Presently, we prepare the Brownian particle in the well at the initial time t_i and ask for the probability P that the particle is still in the well at the final time t_f. Consequently, we obtain this probability by summing the contribution of all paths x_t where the Brownian particle has rested in the past $t < t_i$ at the point[21] x_A

[18] We emphasize that the discontinuous behavior of the Jacobian at $m=0$ will be compensated in the calculation of physical quantities. This is certainly true for the decay rate Γ, which is continuous at $m=0$.

[20] Note the analogy to measuring processes in quantum mechanics. In the simplest case there the event consists of two simple measurements where the first one is usually called a preparation process. See, for instance, ref. 23.

[21] One may ask whether this definition should not be generalized such that the event is also given if x_t passes a suitably chosen environment of x_A. For quasistationary states, however, where $\Gamma \ll \gamma_R$, this generalization produces only a factor which is the same for all contributing paths. Hence, it does not change the expression for the decay rate. The reason for this insensitivity to a precise definition is that for $\Gamma \ll \gamma_R$, we may define kinks and antikinks by

of the metastable minimum and will have returned to x_A at t_f for the rest of its life.

Later, we will find it convenient to make use of a path integral representation where an auxiliary variable y_t appears. Accordingly, we define

$$Q[x_t, y_t] = \text{const} \cdot \exp - \mathscr{B}[x_t, y_t] \tag{3.15a}$$

where

$$\mathscr{B}[x_t, y_t] = \frac{1}{2} \int dt\, dt'\, y_t \tilde{K}_{tt'} y_t + i \int dt\, \xi[x_t]\, y_t \tag{3.16}$$

Obviously,

$$P[x_t] = \int d[y_t]\, Q[x_t, y_t] \tag{3.15b}$$

and we may write for the probability of the event under consideration

$$P(\text{event}) = \int' d[x_t, y_t]\, Q[x_t, y_t] \tag{3.14b}$$

In general, we have to require that y_t vanish for times in the very past and in the distant future.

At this point we recall the path integral representation of ref. 6, for the temporal evolution of the reduced statistical matrix of the Brownian particle, which involves the quantum mechanical action (see footnote 8)

$$\mathscr{B}_{QM}[x_t, y_t] = \frac{1}{2} \int dt\, dt'\, y_t \tilde{K}_{tt'} y_{t'} + i \int dt \left\{ [m\ddot{x}_t + m\gamma \dot{x}_t]\, y_t + V\left(x_t + \frac{1}{2} y_t\right) - V\left(x_t - \frac{1}{2} y_t\right) \right\} \tag{3.17}$$

One recognizes immediately that the action $\mathscr{B}$ of Eq. (3.16) is obtained from $\mathscr{B}_{QM}$ of Eq. (3.17) by the approximation (1.7).

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imposing conditions on their long-time asymptotic behavior. (This is also why the non-Markovian nature of the process does not lead to extra difficulties.)—In this context, we draw attention to the detailed analysis of this problem in ref. 14.—Note that the insensitivity as regards to details is why the event "nondecay" is so popular in decay processes of all kinds. See, for instance, ref. 24, where the instanton approach to quantum tunneling is explained.

In the following, we will evaluate the path integrals asymptotically by restricting our attention to the vicinity of extremal paths. Since $P[x_t]$ and $Q[x_t, y_t]$ are connected by the Gaussian integration (3.15b), the result of the asymptotic theory will be the same for both forms.

Of course, we could have based the functional integral representation immediately on Eq. (3.17) and on its approximate form (3.16). However, it has been our intention to put emphasis on a Langevin equation with (arbitrarily) colored noise, and therefore we have chosen this indirect but instructive approach.

3.4 Reduced Variables

In the case of moderate to large damping, the characteristic rate of the particle in the well [cf. Eq. (1.9)] is given by $\gamma_R = \omega_0^2/\gamma \ll \omega_0$. This suggests that we introduce dimensionless time, frequency, and also temperature variables

$$t^* = t\,\frac{\omega_0^2}{\gamma}, \qquad \omega^* = \omega \cdot \frac{\gamma}{\omega_0^2}$$
$$kT^* = \theta = kT\,\frac{\gamma}{\hbar\omega_0^2}, \qquad \theta_B = \frac{1}{2\pi} \tag{3.18}$$

where the value for the reduced crossover temperature follows from Eq. (1.14). In addition, we define

$$x_t^* = \frac{x_t}{x_B}; \qquad y_t^* = \frac{y_t}{x_B} \tag{3.19}$$

Obviously, $x^* = -1(+1)$ for the particle at the bottom of the well (top of the barrier). Furthermore,

$$\xi_t^* = \frac{1}{m\omega_0^2 x_B}\,\xi_t = M\ddot{x}_t^* + \dot{x}_t^* + \varphi'(x_t^*) \tag{3.20}$$

where

$$M = \omega_0^2/\gamma^2 \tag{3.21}$$

and

$$\varphi(x^*) = \frac{1}{m\omega_0^2 x_B^2}\,V(x) \tag{3.22}$$

For convenience, we omit in the following the asterisk everywhere except in the final result, where $\Gamma = (\omega_0^2/\gamma)\,\Gamma^*$.

Specifically, we have

$$\varphi(x)=\frac{1}{2}x\left(1-\frac{1}{3}x^2\right) \qquad \text{(cp)} \tag{3.23}$$

for the cubic potential (1.11) and

$$\varphi(x)=x\left(1-\frac{1}{2}|x|\right) \qquad \text{(ppp)} \tag{3.24}$$

for the piecewise parabolic potential (1.12). Clearly, $\varphi'(\pm 1)=0$ and $\varphi''(\pm 1)=\mp 1$ in both cases.

Of interest will be some properties of the deterministic motion $\xi_t=0$ for small displacements from the equilibrium points $x=\pm 1$. The general solution is

$$\begin{aligned} &c_1^+\exp(-\lambda_1^+ t)+c_2^+\exp(\lambda_2^+ t); \qquad x=+1\\ &c_1^-\exp(-\lambda_1^- t)+c_2^-\exp(-\lambda_2^- t); \qquad x=-1 \end{aligned} \tag{3.25}$$

where

$$\begin{aligned} \lambda_{\frac{1}{2}}^+ &=\frac{1}{2M}\left[(1+4M)^{1/2}\pm 1\right]\\ \lambda_{\frac{1}{2}}^- &=\frac{1}{2M}\left[1\pm(1-4M)^{1/2}\right] \end{aligned} \tag{3.26}$$

Note the relations

$$\begin{aligned} &\lambda_1^\pm>\lambda_2^\pm>0; \qquad \lambda_1^\pm\cdot\lambda_2^\pm=\frac{1}{M}\\ &\lambda_1^+-\lambda_2^+=\lambda_1^-+\lambda_2^- \end{aligned} \tag{3.27}$$

Of importance will also be the dimensionless quantity

$$\mathscr{U}=\frac{m\gamma x_B^2}{\hbar}=\frac{\gamma}{\hbar\omega_0^2}V_B\cdot u_0, \qquad u_0=\begin{cases}3/2 & \text{(cp)}\\ 1 & \text{(ppp)}\end{cases} \tag{3.28}$$

which allows us to write in terms of the new quantities (asterisks omitted again)

$$\begin{aligned} Q[x_t,y_t]&=\text{const}\cdot\exp-\mathscr{U}\mathscr{B}[x_t,y_t]\\ \mathscr{B}[x_t,y_t]&=\frac{1}{2}\int dt\,dt'\,y_t\tilde{K}_{tt'}y_{t'}+i\int dt\,\xi_t y_t\\ \xi_t&=M\ddot{x}_t+\dot{x}_t+\varphi'(x_t)\\ \tilde{K}_\omega&=\omega\coth\frac{\omega}{2\theta} \end{aligned} \tag{3.29}$$

Note also that in these reduced variables,

$$P[x_t] = \text{const} \cdot \exp - \mathcal{U}\mathcal{A}(x_t)$$
$$\mathcal{A}[x_t] = \frac{1}{2}\int dt\, dt'\, \xi_t \tilde{N}_{tt'} \xi_{t'} \tag{3.30}$$
$$\tilde{N}_\omega = \frac{1}{\omega} \tanh \frac{\omega}{2\theta}$$

3.5. Extremal Paths

We expect that typical values of the (reduced) frequency are of the order $\lambda_2^\pm \sim 1$. Similarly, we estimate that ξ_t and $\int dt\, \xi_t$ are also of the order 1. Therefore, we have $\tilde{N}_\omega \sim \min(\theta^{-1}, |\omega|^{-1} \sim 1)$, and it follows that $\mathcal{U}\mathcal{A}[x_t]$ as given in Eq. (3.30) is of the order V_B/kT_N, where T_N has been introduced in Eq. (1.13). We conclude that the condition $V_B \gg kT_N$ as stated in Eq. (1.13) guarantees that the main contribution to the functional integrals comes from small regions in the vicinity of extremal paths (x_t^p, y_t^p) which minimize the action $\mathcal{A}$. Note that these paths are also extremal paths of the extended action $\mathcal{B}$. Therefore, we look first for solutions of

$$\frac{\delta\mathcal{B}}{\delta x_t} = 0; \qquad \frac{\delta\mathcal{B}}{\delta y_t} = 0; \qquad (x_y, y_t) = (x_t^p, y_t^p) \tag{3.31}$$

The ensuing values for the action will be denoted as follows:

$$\mathcal{A}^p = \mathcal{B}^p = \mathcal{A}[x_t^p] = \mathcal{B}[x_t^p, y_t^p] \tag{3.32}$$

Explicitly, Eq. (3.31) corresponds to the following equation of motion:

$$[M\partial_t^2 - \partial_t + \varphi''(x_t^p)]\, \phi_t^p = 0 \tag{3.33a}$$
$$[M\ddot{x}_t^p + \dot{x}_t^p + \varphi'(x_t^p)] + \int dt'\, \tilde{K}_{tt'} \phi_{t'}^p = 0 \tag{3.33b}$$
$$\phi_t^p := -iy_t^p \tag{3.33c}$$

Above we have introduced the notation $\phi_t = -iy_t$ in order to avoid imaginary quantities.

First of all, note the trivial solutions $x_t^\pm = \pm 1$ and $\phi_t^\pm = 0$ $(p = \pm)$. Most important are kink- and antikink-like solutions $(p = k, a)$ which connect the two trivial solutions. For illustration, see Fig. 7, where a kink (at t_k) is followed by an antikink (at t_a) such that the combined object $(p = c)$

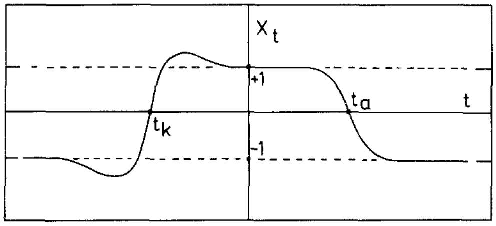

Fig. 7. Kink followed by and antikink centered at t_k and t_a, respectively. Together they form a combined object which contributes to the probability P that the Brownian particle remains in the metastable well.

is a path which contributes to the probability of the particle remaining in the metastable state as explained in Section 3.3.

Consider now an *isolated antikink* centered at $t_a = 0$. It obeys the deterministic equation $\phi_t^a = \xi_t^a = 0$; explicitly,

$$M\ddot{x}_t^a + \dot{x}_t^a + \varphi'(x_t^a) = 0 \tag{3.34}$$

For illustration, we give the solution of the above equation in the limit $M = 0$:

$$x_t^a = \begin{cases} -\tanh t/2 & \text{(cp)} \\ -(1 - e^{-|t|}) \operatorname{sgn} t & \text{(ppp)} \end{cases} \tag{3.35}$$

A more detailed description is given in Sections 4 and 5 for ppp and cp, respectively.

Consider next an *isolated kink* centered at $t_k = 0$. In general its functional form is not so easy to obtain and its discussion will be deferred to Sections 4 (ppp) and 5 (cp). However, in the white noise limit (WN) where $\tilde{K}_{tt'} = 2\theta\delta(t - t')$ one can show[22] that the kink is a time-reversed antikink such that, for arbitrary M,

$$x_t^k = x_{-t}^a; \qquad \phi_t^k = \frac{1}{\theta} \dot{x}_{-t}^a \qquad \text{(WN)} \tag{3.36}$$

Note that the antikink, as well as the kink in the white noise limit, approach their asymtotic values exponentially fast. In contrast to this behavior, the kink shows only an algebraic asymptotics in the blue noise limit (BN), that is, for $\theta = 0$.

Calculating the extremal values of the action, we find that

$$\mathscr{B}^k = \frac{1}{2} \int dt\, dt'\, \phi_t^k \tilde{K}_{tt'} \phi_{t'}^k; \qquad \mathscr{B}^a = 0 \tag{3.37}$$

[22] Note that $[M\partial_t^2 + \partial_t + \varphi''(x_t^a)]\, \dot{x}_t^a = 0$.

Therefore, we have in the white noise limit

$$\begin{aligned} \mathscr{B}^k &= \frac{1}{\theta} \int dt (\dot{x}^a_{-t})^2 = \frac{1}{u_0 \theta} \qquad \text{(WN)} \\ \mathscr{U}\mathscr{B}^k &= \frac{V_B}{kT} \end{aligned} \tag{3.38}$$

For later reference, we define here the normalization

$$\begin{aligned} \beta_p^2 &= \int dt (|\dot{x}_t^p|^2 + |\dot{y}_t^p|^2); \qquad p = a, k,... \\ \beta_a^2 &= \frac{1}{u_0} \end{aligned} \tag{3.39}$$

3.6. Kink-Antikink Interaction

Approximately, the combined object shown in Fig. 7 can be represented by

$$\begin{aligned} x_t^c &\simeq x^k_{t-t_k} + x^a_{t-t_a} - 1 \\ \phi_t^c &\simeq \phi^k_{t-t_k} \end{aligned} \tag{3.40}$$

provided that kink and antikink are well separated, $(t_a - t_k) \gg 1$. For large separation, we also have

$$\mathscr{B}^c = \mathscr{B}^k \tag{3.41}$$

However, there is, strictly speaking, no solution of Eq. (3.33) resembling a combined object, due to the attractive interaction between kink and antikink. In order to overcome this difficulty, we add to the action a source term

$$\begin{aligned} \mathscr{C}[x_t, y_t] &= \mathscr{B}[x_t, y_t] + \mathscr{C}_{\text{source}}[x_t] \\ \mathscr{C}_{\text{source}} &= -b \int dt [1 + x_t] \end{aligned} \tag{3.42}$$

which is meant to pull kink and antikink apart. Indeed, there exists now an extremal path (x_t^c, ϕ_t^c) which minimizes $\mathscr{C}$ and which is a function of b. Thus,

$$\mathscr{C}^c = \mathscr{C}(b) \tag{3.43a}$$

such that[23]

$$\frac{\partial \mathscr{C}^c}{\partial b} = -2t_0; \qquad t_0 = t_a - t_k \tag{3.43b}$$

This means that in the Legendre transform

$$\mathscr{B}^c(t_0) = \mathscr{C}^c - b\,\frac{\partial \mathscr{C}^c}{\partial b} = \mathscr{B}^c + \mathscr{B}_{\rm int}(t_0) \tag{3.44}$$

a term $\mathscr{B}_{\rm int}(t_0)$ appears which has the meaning of a kink–antikink interaction. Later, we will find for ppp that

$$\begin{aligned} \mathscr{B}_{\rm int}(t_0) &= -g\mathscr{B}^k \exp(-\lambda_2^+ t_0) \qquad \text{(ppp)} \\ g &= 2 - \lambda_2^+ = (\lambda_1^+ + \lambda_2^+)/\lambda_1^+ \end{aligned} \tag{3.45}$$

provided that $\lambda_2^+ t_0 \gg 1$. One may be surprised about the exponential decay above, since the kink asymptotics is only algebraic for $\theta = 0$. Nevertheless, we have reason to assume that the form (3.45) is also valid for cp, where g is expected to be also a factor of order unity.

3.7. Gaussian Fluctuations about Extremal Paths

Let us introduce the small quantities

$$\begin{aligned} \zeta_t &= x_t - x_t^p \\ \eta_t &= \phi_t - \phi_t^p \end{aligned} \tag{3.46}$$

Then, we have through second order

$$\begin{aligned} \mathscr{B}[x_t, y_t] &= \mathscr{B}^p + \mathscr{L}^p[\zeta_t, \eta_t] \\ \mathscr{L}^p &= -\frac{1}{2}\int dt\, dt' (\zeta_t, \eta_t)\, \mathscr{H}^p_{tt'} \begin{pmatrix} \zeta_{t'} \\ \eta_{t'} \end{pmatrix} \end{aligned} \tag{3.47}$$

($\mathscr{B}$ may also be replaced by $\mathscr{C}$), where the fluctuation operator $\mathscr{H}^p$ is given by

$$\mathscr{H}^p_{tt'} = \delta(t-t')\begin{pmatrix} \phi_t^p \varphi'''(x_t^p) & \bar{h}_t^p \\ h_t^p & 0 \end{pmatrix} + \tilde{K}_{tt'} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.48}$$

[23] Except for small corrections of order $(\lambda_2^\pm)^{-1}$ which are due to the fact that the "center of mass" of kink and antikink does not coincide exactly with t_k and t_a, respectively.

Above, h_t^p is the differential operator

$$h_t^p = M\partial_t^2 + \partial_t + \varphi''(x_t^p) \tag{3.49}$$

and its adjoint operator $\bar{h}_t^p$ is obtained by letting $\partial_t \to -\partial_t$. We also define the retarded Green's function

$$\begin{aligned} h_t^p G_{tt'}^p &= \delta(t-t') \\ G_{tt'}^p &= 0 \qquad \text{for} \quad t < t' \end{aligned} \tag{3.50}$$

Note that $G_{tt'}^p$ is bounded for $|t|, |t'| \to \infty$ if $\varphi''(x_{\pm\infty}^p) > 0$, i.e., when x_t^p approaches a local equilibrium position in distant times (compare $G_{tt'}^p$ with the Green's function $G_{tt'}^0$ introduced in Section 3.2). In fact, this is the case for $p = -, c$.

Let us now consider specifically the combined object where $p = c$. As an important point here, note that the fluctuation operator $\mathscr{H}^c$ has, in addition to the eigenvalue zero of the translational mode, an exponentially, $\lesssim \exp(-\lambda_2^+ t_0)$, small eigenvalue of the breather mode which is a motion of kink and antikink against each other. Clearly, the Gaussian aproximation breaks down in these two cases; instead, we have to integrate with respect to the translational (t_T) and the breather (t_B) time coordinate, including a proper weight, say w_{TB}. On the other hand, these two coordinates are just linear combinations of kink and antikink positions t_k and t_a, respectively. In the last case, it is known[21,24] that the weight is just $\beta_k \beta_a / 2\pi$, where the normalization β_p is defined by Eq. (3.39).

Considering, then, a finite time interval, say $0 < t_k < t_a < \bar{t}$, and normalizing the path integral by division with the Gaussian fluctuations about the stable trivial path where $(x_t^-, \phi_t^-) = (-1, 0)$, we obtain the contribution $P_1(\bar{t})$ of *one* combined object to the probability $P(\bar{t})$ of the particle to remain in the well:

$$P_1(\bar{t}) = \frac{\beta_a \beta_k}{2\pi} \mathscr{U} \int_0^{\bar{t}} dt_k \int_{t_k}^{\bar{t}} dt_a \left(\frac{\det \mathscr{H}^-}{\det'' \mathscr{H}^c} \right)^{1/2} \exp[-\mathscr{U}\mathscr{B}^c(t_a - t_k)] \tag{3.51}$$

where $\mathscr{B}^c(t_0)$ is given in Eq. (3.44). The double prime in $\det'' \mathscr{H}^c$ means that the translation and breather eigenvalues have to be omitted; note that

$$(\det \mathscr{U}\mathscr{H}^- / \det'' \mathscr{U}\mathscr{H}^c)^{1/2} = \mathscr{U}(\det \mathscr{H}^- / \det'' \mathscr{H}^c)^{1/2}$$

Assuming the interaction between different combined objects to be negligible, we have in the usual dilute gas approximation for n combined objects $P_n(\bar{t}) = [P_1(\bar{t})]^n / n!$; thus, after summation with respect to n ($n = 0$

includes the trivial path $p=-$), we obtain the total probability of the particle to remain in the well:

$$P(\bar{t})=\exp P_1(\bar{t}) \tag{3.52}$$

3.8. Properties of the Fluctuation Determinants

Roughly, one may distinguish three different contributions to $\det'' \mathscr{H}^c$ of Eq. (3.51). Two contributions are connected with the transitions between the trivial paths as represented by kink and antikink. The third contribution is connected with the sojourn at $x=+1$. We can understand the last contribution by calculating $\det \mathscr{H}^+/\det \mathscr{H}^-$ for, say, a time t_0. Assuming periodic boundary conditions and taking frequencies $\omega_n=2\pi n/t_0$, we find that (for $\lambda_2^+ t_0 \to \infty$)

$$\frac{\det \mathscr{H}^+}{\det \mathscr{H}^-}=\prod_n \left|\frac{-M\omega_n^2+i\omega_n-1}{-M\omega_n^2+i\omega_n+1}\right|^2 \to \exp(2\lambda_2^+ t_0) \tag{3.53}$$

provided that $M\neq 0$. On the other hand,

$$\frac{\det \mathscr{H}^+}{\det \mathscr{H}^-}=\prod_n \left|\frac{i\omega_n-1}{i\omega_n+1}\right|^2 \to 1 \qquad (M=0) \tag{3.54}$$

We emphasize that *this discontinuous behavior is exactly compensated by a discontinuity* (see footnote 18) of the Jacobian $d[\xi_t]/d[x_t]$ as discussed in Section 3.2 [cf. Eqs. (3.11) and (3.13)].

In the following, we will always assume that $M\neq 0$, though possibly small, $M\to 0$. Therefore, we find it convenient to introduce the quantity

$$\mathscr{Y}=\left(\frac{\det'' \mathscr{H}^c}{\det \mathscr{H}^-}\right)^{1/2} \exp(-\lambda_2^+ t_0) \tag{3.55}$$

which we expect to become independent of t_0 for $\lambda_2^+ t_0\to\infty$.

From the description of the various contributions given at the beginning of this section, it seems intuitively clear that the fluctuations about the combined object can be written as a product of the fluctuations about the kink and antikink. In fact, we have

$$\frac{\det'' \mathscr{H}^c}{\det \mathscr{H}^-}=\frac{\det' \mathscr{H}^k}{\det \mathscr{H}^-}\,\frac{\det' \mathscr{H}^a}{\det \mathscr{H}^-}\left(\frac{\lambda_1^+ + \lambda_2^+}{\lambda_1^- + \lambda_2^-}\right)^2 \tag{3.56}$$

where the fluctuation spectrum should be calculated for zero boundary conditions[24] and where the additional factor corrects for the fact that in the

[24] The ratio of the fluctuation spectra depends on the boundary condition whenever the fluctuation operators in numerator and demoninator differ near the boundary. For instance, $\mathscr{H}^k$ and $\mathscr{H}^-$ are different at the boundary to the right of t_k.

above decomposition, integration over intermediate coordinate and velocity has to be included.

3.9. The Decay Rate

From Eqs. (3.44), (3.51), and (3.55), we obtain

$$P_1(\bar{t}) = \frac{\beta_a \beta_k}{2\pi} \frac{\mathscr{U}}{\mathscr{Y}} e^{-\mathscr{U}\mathscr{B}^k} \int_0^{\bar{t}} dt_k \int_0^{\bar{t}-t_k} dt_0 \, e^{-W(t_0)} \tag{3.57}$$

where

$$W(t_0) = \lambda_2^+ t_0 - \mathscr{U}\mathscr{B}_{\text{int}}(t_0) \tag{3.58}$$

The important point is that $W(t_0)$ includes the long-ranged (fluctuation-induced) term $\lambda_2^+ t_0$ which suppress sizable contributions from large t_0 in the integral above. Therefore, it seems that the short range part $\mathscr{U}\mathscr{B}_{\text{int}}$ plays a crucial role in the present problem. Unfortunately, $\mathscr{B}_{\text{int}}$ is smallest for $t_0 = 0$, where the concept of a combined object, consisting of kink and antikink, breaks down.

This problem has already been discussed for the white noise decay problem in ref. 14. As a remedy,[25] it has been proposed there to shift the path of the t_0 integration from $\text{Im}\, t_0 = 0$ to $\text{Im}\, t_0 = i\pi/\lambda_2^+$. Then, it is possible to extend the limits of the t_0 integration to $\text{Re}\, t_0 = \pm\infty$. As a result, we obtain

$$P_1 = -\Gamma \bar{t}; \qquad P(\bar{t}) = e^{-\Gamma \bar{t}} \tag{3.59}$$

and we recognize that the decay rate Γ is of the standard form (1.10). Specifically

$$\begin{aligned} a &= (\omega_0 \beta_a \beta_k / \gamma) / (g \lambda_2^+ \mathscr{Y} \mathscr{B}^k) \\ b &= \mathscr{U}\mathscr{B}^k \end{aligned} \tag{3.60}$$

where we have multiplied the prefactor by the factor ω_0^2/γ in order to return to ordinary units.

[25] In the context of quantum mechanics, a formally quite similar problem is discussed in ref. 25. There is proposed to change the sign of the "coupling constant" $\mathscr{U}$. Hence $\mathscr{U} \to -\mathscr{U}$ for the integration; and then $-\mathscr{U} \to \mathscr{U}$. One can see easily that this procedure leads to the same result as the shift of the integration contour. A completely different method has been developed by Weiss and Theisen.[26] Again, the final result is the same.

4. THE PIECEWISE PARABOLIC POTENTIAL

4.1. Self-Consistent Solution

Consider the form (3.24) for the piecewise parabolic potential (ppp) and observe its derivatives

$$\begin{aligned} \varphi'(x) &= 1 - x \operatorname{sgn} x \\ \varphi''(x) &= -\operatorname{sgn} x \\ \varphi'''(x) &= -2\delta(x) \end{aligned} \tag{4.1}$$

This means that the nonlinearity of the potential appears only at those times t_l where x_t changes its sign; i.e., where $x_{t_l} = 0$. Therefore, we may solve the equation of motion (3.33) for the extremal path as follows. We assume that the times t_l are known; then we rewrite the equation of motion in a linear form and find its general solution. Eventually, we determine t_l from the condition of self-consistency $x_{t_l} = 0$. Clearly, this strategy works best if there are only few zeros of x_t; consequently, we restrict our attention to the overdamped case $M < 1/4$, where one expects that the equation $x_t = 0$ has only one solution for the kink and the antikink, and two solutions of the combined object

For later convenience, let us introduce the notation

$$\Theta_t^p = \varphi''(x_t^p) = -\operatorname{sgn} x_t^p \tag{4.2}$$

which enters in the definition (3.49) of the operator h_t^p.

4.2 The Antikink

We choose $t_a = 0$, whence it follows that $x_t^a \gtrless 0$ for $t \lessgtr 0$ and $\Theta_t^a = \operatorname{sgn} t$. Therefore, $\varphi'(x_t^a) = 1 + x_t^a \Theta_t^a$, and the equation for the antikink (3.34) assumes the form

$$h_t^a x_t^a + 1 = 0 \tag{4.3}$$

The bounded solution of the linear equation (4.3) is

$$x_t^a = \begin{cases} 1 - e^{\lambda_2^+ t} & t < 0 \\ -1 + B_e{}^{-\lambda_1^- t} + B_2 e^{-\lambda_2^- t}, & t > 0 \end{cases} \tag{4.4}$$

where $\lambda_l^\pm$ is defined by Eq. (3.26) and where the coefficients B_l are such that x_t^a and $\dot{x}_t^a$ are continuous. In the limit $M \to 0$ where $\lambda_2^\pm \to 1$ and

$\lambda_1^- \to \infty$, we confirm the ppp part of Eq. (3.35). Note that the self-consistency $x_{t=0}^a = 0$ is guaranteed by the ansatz (4.4).

Later, the antikink translational mode

$$\psi_t^a = -\dot{x}_t^a; \qquad \dot{\psi}_0^a = \lambda_2^+ \tag{4.5}$$

will be of importance. In the limit $M \to 0$, we find easily from Eq. (3.35) that $\psi_t^a = e^{-|t|}$. For easy reference we give the general expression in terms of the Fourier transform, where

$$\psi_\omega^a = \frac{2M\lambda_2^+(i\omega - \lambda_1^+)}{(M\omega^2 + i\omega + 1)(M\omega^2 + i\omega - 1)} \tag{4.6}$$

4.3. The Kink

We choose $t_k = 0$ and obtain $x_t^k \gtrless 0$ for $t \gtrless 0$. Therefore, $\Theta_t^k = -\text{sgn}\, t$ and $\varphi'(x_t^k) = 1 + x_t^k \Theta_t^k$; this allows us to write Eq. (3.33a) as follows:

$$\bar{h}_t^k \phi_t^k = 0 \tag{4.7}$$

Comparing Eq. (4.7) with the time derivative (see footnote 22) of Eq. (4.3), we conclude that

$$\phi_t^k = C \cdot \psi_{-t}^a \tag{4.8}$$

and inserting this relation in Eq. (3.33b), we obtain

$$h_t^k x_t^k + 1 + C \int dt'\, \tilde{K}_{tt'} \psi_{-t'}^a = 0 \tag{4.9}$$

The constant C can be determined as follows. We multiply Eq. (4.9) from the left by ψ_{-t}^a and integrate with respect to t. Taking into account that $\bar{h}_t^k \psi_{-t}^a = 0$ and that $\int dt\, \psi_t^a = 2$, we arrive at

$$C = \mathscr{B}^k = 2\left(\int \frac{d\omega}{2\pi} |\psi_\omega^a|^2 \tilde{K}_\omega\right)^{-1} \tag{4.10}$$

For sake of completeness we have added the relation $C = \mathscr{B}^k$, which follows from Eqs. (3.37) and (4.8).

Specifically, in the white noise limit $\theta \gg \theta_B$, we obtain

$$C = \mathscr{B}^k = \frac{1}{\theta}\left[1 - \frac{(\lambda_2^+)^2}{12\theta^2} + \cdots\right] \qquad \text{(WN)} \tag{4.11}$$

On the other hand, we find for blue noise $(\theta \ll \theta_B)$ the result that $\mathscr{B}^k \to \pi$ for $M \to 0$; and that $\mathscr{B}^k$ increases slowly with increasing M to the value

1.07π for $M \to 1/4$. For finite θ, we have calculated $\mathscr{B}^k$ by numerical integration. The result is shown in Fig. 8 (Section 4.5) for $M = 1/9$.

It is important to note that the condition (4.10) for C guarantees (i) that x_t^k is bounded and (ii) that $x_{t=0}^k = 0$. This can be shown as follows. First we recall the definition (3.50) of the Green's function $G_{tt'}^p$. For illustration, consider the limit $M = 0$, where

$$\begin{aligned} G_{tt'}^p &= \theta(t-t')\exp(-h_t^p + h_{t'}^p) \qquad (M=0) \\ h_t^p &= \int_0^t dt'\, \Theta_{t'}^p \end{aligned} \tag{4.12}$$

If convenient, we also make use of the full Fourier transform

$$G_{\omega\omega'}^p = \int dt\, dt'\, e^{i\omega t - i\omega' t} G_{tt'}^p \tag{4.13}$$

as well as the partial transforms $G_{\omega t'}^p$ and $G_{t\omega'}^p$ (note that not all combinations exist; however, we will need only those that do exist). Then we may write the solution of Eq. (4.9) as follows:

$$x_t^k = x_{-t}^a + \int \frac{d\omega}{2\pi} G_{t\omega}^k [2 - C\tilde{K}_\omega]\, \psi_{-\omega}^a \tag{4.14}$$

In the white noise (WN) limit, we have $C \cdot \tilde{K}_\omega = \theta^{-1} \cdot 2\theta = 2$ and it follows that $x_t^k = x_{-t}^a$; this result has already been noted in Eq. (3.36).

For blue noise (BN) and in the limit $M = 0$ where $C = \pi$, we obtain from Eqs. (4.6), (4.12), and (4.14) as well as (3.35) [$E_1(t)$ and $Ei(t)$ are standard exponential integrals[27]]

$$\begin{aligned} x_t^k &= \operatorname{sgn} t\, [(\tfrac{1}{2} + |t|)\, e^{|t|} E_1(|t|) + \tfrac{1}{2} e^{-|t|} Ei(|t|)] \\ \phi_t^k &= \pi e^{-|t|} \qquad (\text{BN}; M=0) \\ C &= \mathscr{B}^k = \pi \\ x_t^a &= -\operatorname{sgn} t\, (1 - e^{-|t|}) \end{aligned} \tag{4.15}$$

(For the sake of quick reference, we have repeated some relations.) Schematically, these types of kink and antikink are shown in the left and right parts of Fig. 7, respectively. In contrast to the white noise limit, where $x_t^k = x_{-t}^a$, the kink shows a kind of overshooting. It appears that the particle first moves in the "wrong" direction "to gain speed" for climbing up the barrier. From a formal point of view, this behavior is due to the long-time correlation (1.6) of the stochastic force. Explicitly, it follows from

Eq. (4.15) that $x_t^k \to -2t \ln|t|$ and $x_t^k \to \pm(1+2/t^2)$ for $t \to 0$ and $t \to \pm\infty$, respectively; the latter relation confirms a previous statement on the slow algebraic asymptotics.

Of importance will be the velocity $v_p = \dot{x}^p_{t=0}$. From Eqs. (4.5) and (4.14), we obtain

$$\begin{aligned} v_a &= -\lambda_2^+ \\ v_k &= \lambda_2^+ + \int \frac{d\omega}{2\pi} \frac{i\omega}{M\omega^2 + i\omega - 1} (2 - C\tilde{K}_\omega)\, \psi^a_{-\omega} \end{aligned} \tag{4.16a}$$

where ψ^a_ω and C are given by Eqs. (4.6) and (4.10). In particular, we have for $M \to 0$

$$v_k = \begin{cases} 1 & \text{(WN)} \\ \left(2 \ln \dfrac{1}{M}\right)^{-1} & \text{(BN)} \end{cases} \tag{4.16b}$$

4.4. The Combined Object

Let t_k and t_a be the positions of kink and antikink (see Fig. 7); then $\operatorname{sgn} x_t^c = \pm 1$ for $(t-t_k)(t-t_a) \lessgtr 1$. Following Eq. (4.2), we define the quantity

$$\Theta_t^c(t_k, t_a) = \varphi''(x_t^c) = \operatorname{sgn}[(t-t_k)(t-t_a)] \tag{4.17a}$$

and include it in the definition (3.49) of the operator h_t^c.

For convenience, we will work in this section with the symmetric choice $t_k = -t_0/2$, $t_a = +t_0/2$. Then (we obtain the general expression if we substitute $t \to t - (t_k + t_a)/2$, $t_0 \to t_a - t_k$ in the symmetrized form)

$$\Theta_t^c = \operatorname{sgn}(t^2 - t_0^2/4) \tag{4.17b}$$

With this choice, we have $\bar{h}_t^c = h^c_{-t}$.

According to the program outlined in Section 3.6, we add a source term to the action and replace $\mathscr{B}[x_t, y_t]$ by $\mathscr{C}[x_t, y_t]$ as shown in Eq. (3.42). As a consequence, Eq. (3.33a) defining the extremal path ϕ_t^c is replaced by

$$\bar{h}_t^c \phi_t^c = -b \tag{4.18}$$

We introduce now the Green's function $G^c_{tt'}$, according to Eq. (3.50). In view of the symmetry $\bar{h}_t^c = h^c_{-t}$, we have $\bar{G}^c_{tt'} = G^c_{-t,-t'}$. Thus, the solution of Eq. (4.18) can be written as

$$\phi_t^c = -G^c_{-t,\omega'=0} \cdot b \tag{4.19}$$

Inserting this relation in Eq. (3.33b)

$$h_t^c x_t^c + 1 + \int dt'\, \tilde{K}_{tt'} \phi_{t'}^c = 0 \tag{4.20}$$

we find that its solution is given by

$$x_t^c = \int \frac{d\omega}{2\pi} G_{t,\omega}^c \left[b \cdot \tilde{K}_\omega G_{-\omega,\omega'=0}^c - 2\pi\delta(\omega) \right] \tag{4.21}$$

It requires some calculation to show that for any $\tilde{K}_\omega = \tilde{K}_{-\omega}$, the self-consistency conditions $x_{t_0/2}^c = x_{-t_0/2}^c = 0$ are satisfied simultaneously if we choose $b = b(t_0)$ properly. Specifically, we have for blue noise and for large damping

$$b = -\tfrac{1}{2}\pi e^{-t_0}; \qquad t_0 \gg 1 \qquad (\theta = M = 0) \tag{4.22}$$

Concerning some details, note that in the limit of large damping,

$$-\frac{1}{b}\phi_t^c = \begin{cases} 1 + 2e^{(t+t_0/2)}(e^{t_0} - 1), & t < -t_0/2 \\ -1 + 2e^{-(t-t_0/2)} & |t| < t_0/2 \\ 1 & t > t_0/2 \end{cases} \qquad (M=0) \tag{4.23}$$

Therefore, we have in leading order

$$\begin{aligned} x_t^c &= x_{t+t_0/2}^k; \qquad \phi_t^c = \phi_{t+t_0/2}^k \qquad (t+t_0/2) \sim 0 \\ x_t^c &= x_{t-t_0/2}^a; \qquad \phi_t^c = O(e^{-t_0}) \qquad (t-t_0/2) \sim 0 \end{aligned} \qquad (M=0) \tag{4.24}$$

where x_t^k, ϕ_t^k, and x_t^a are given in (4.15). For arbitrary damping $M \neq 0$, the above relation can easily be generalized; for later reference, we need the following relations:

$$\begin{aligned} \phi_t^c &= \phi_{t+t_0/2}^k = \mathscr{B}^k \psi_{-(t+t_0/2)}^a \qquad (t+t_0/2) \sim 0 \\ \phi_{-t_0/2}^c &= \lambda_2^+ \mathscr{B}^k \\ G_{t_0/2,\,-t_0/2}^c &= \frac{\lambda_2^+}{g} e^{\lambda_2^+ t_0} \end{aligned} \tag{4.25}$$

where g has been introduced in Eq. (3.45).

Concerning the action $\mathscr{C}^c$ as defined by Eq. (3.43), we should observe that the applied force shifts the trivial path according to

$$x_t^- = -1 - b \cdot \tilde{K}_{\omega=0} \tag{4.26}$$

Correspondingly, there is an "infinite" change of the action $\propto b^2$, which we subtract.

For the sake of simplicitly, let us discuss the details in the case $M=\theta=0$, where Eq. (4.22) is valid. Eliminating t_0, we obtain

$$\mathscr{C}^c(b)=\pi+2b\left(\ln\frac{\pi}{-2b}+1\right) \tag{4.27a}$$

Performing the Legendre transform according to Eq. (3.44), we arrive at

$$\mathscr{B}^c(t_0)=\mathscr{B}^k(1-e^{-t_0}) \tag{4.27b}$$

where presently $\mathscr{B}^k=\pi$.

For arbitrary M, one finds the result already given by Eq. (3.45). Note that the kink–antikink interaction depends on the color of the noise only through $\mathscr{B}^k$.

4.5. Fluctuation Determinant of the Combined Object

Ignoring for a moment the existence of the zero and close-to-zero eigenvalues associated with the translational and breather modes, we may write

$$(\det \mathscr{H}^c)^{-1/2}=\text{const}\cdot\int d[\zeta_t,\eta_t]\exp-\mathscr{L}^c[\zeta_t,\eta_t] \tag{4.28}$$

where the operator $\mathscr{H}^c$ and the associated quadratic form $\mathscr{L}^c$ are defined by Eqs. (3.48) and (3.47). Consider now the quantity $\phi_t^c\cdot\varphi'''(x_t^c)$ which appears in the definition of $\mathscr{H}^c$. Assuming that the kink and antikink are located at t_k and t_a, respectively, and considering Eq. (4.1), we obtain

$$\begin{gathered}\phi_t^c\cdot\varphi'''(x_t^c)=-\sum_{p=k,a}w_p\delta(t-t_p)\\ w_p=2\phi_{t_p}^c/|v_p|\end{gathered} \tag{4.29}$$

where the velocities v_p are given by Eq. (4.16).

The contributions of the translational and breather modes can be eliminated elegantly as follows (see ref. 28; also see ref. 29, Appendix B). Let us recall first that the probability of the event under consideration is originally given by the path integral (3.14b). Now, we choose for each configuration $(x_t, y_t=i\phi_t)$ of the integration field an extremal path (x_t^c, ϕ_t^c) in the form of a combined object (with $x_{t_k}=x_{t_a}=0$) such that their distance $\mathscr{D}(x_t, \phi_t\,|\,x_t^c, \phi_t^c)$ is as small as possible. Though there exists a standard expression for this distance it is by no means compulsatory. We take advantage of this freedom[26] and choose

$$\mathscr{D}(x_t, \phi_t \,|\, x_t^c, \phi_t^c) = \int dt[x_t - \Theta_t^c(t_k, t_a)]^2 \tag{4.30}$$

where $\Theta_t^c = \varphi''(x_t^c)$ is given by Eq. (4.17a).

Observe now that

$$\frac{d}{dt_k} \Theta_t^c(t_k, t_a) = 2\delta(t - t_k) \operatorname{sgn}(t_a - t_k)$$
$$\int dt_k \, 2 \, |\dot{x}_{t_k}| \, \delta\left(\frac{d}{dt_k} \int dt \, x_t \Theta_t^c\right) = 1 \tag{4.31}$$

and that there is a similar relation with t_k and t_a interchanged. We recognize clearly that $\mathscr{D}$ is minimal if $x_{t_k} = x_{t_a} = 0$. Arguing now in a standard way,[(28,29)] we conclude that

$$\frac{\beta_k \beta_a}{2\pi} \left(\frac{\det \mathscr{H}^-}{\det'' \mathscr{H}^c}\right)^{1/2} = v_a v_k \mathscr{R} \tag{4.32}$$

where

$$\mathscr{R} = N^{-1} \int d[\zeta_t, \eta_t] \, \delta(\zeta_{t_k}) \, \delta(\zeta_{t_a}) \exp - \mathscr{L}^c[\zeta_t, \eta_t]$$
$$N = \int d[\zeta_t, \eta_t] \exp - \mathscr{L}^-[\zeta_t, \eta_t] \tag{4.33}$$

Note that above we replaced $\dot{x}_{t_p}$ by $\dot{x}_{t_p}^c = v_p$.

The advantage of the choice (4.30) is that it produces the two δ-functions in Eq. (4.33), which allows us to discard the contribution

$$\int dt \, \zeta_t^2 \phi_t^c \varphi'''(x_t^c) = -\sum_p w_p \zeta_{t_p}^2 \tag{4.34}$$

[see Eq. (4.29)] to $\mathscr{L}^c[\zeta_t, \eta_t]$.

Let us represent the δ-functions in Eq. (4.33) by two Fourier integrals with respect to $\boldsymbol{\rho} = (\rho_k, \rho_a)$. Then

$$\mathscr{R} = \int \frac{d^2\rho}{(2\pi)^2} \exp - \frac{1}{2} \boldsymbol{\rho} \mathscr{M} \boldsymbol{\rho} \tag{4.35}$$

[26] The only condition is that the prescription fixes the zero-frequency modes. This is guaranteed with the present choice, since $\int dt \, \dot{\Theta}_t^p \dot{x}_t^p \neq 0$ for $p = k, a, c$.

where the matrix $\mathscr{M}$ is defined by

$$\exp -\frac{1}{2}\boldsymbol{\rho}\mathscr{M}\boldsymbol{\rho} = N^{-1}\int d[\zeta_t, \eta_t] \exp - \tilde{\mathscr{L}}^c[\zeta_t, \eta_t]$$
$$\tilde{\mathscr{L}}^c = \int dt\, \zeta_t \left[h_t^c \eta_t + i \sum_p \rho_p \delta(t - t_p) \right] - \frac{1}{2}\int dt\, dt'\, \eta_t \tilde{K}_{tt'} \eta_{t'} \tag{4.36}$$

Since $\tilde{\mathscr{L}}^c$ depends on ζ_t only linearly, the $[\zeta_t]$ integration[27] produces the δ-functional

$$\delta \left[\bar{h}_t^c \eta_t + i \sum_p \rho_p \delta(t - t_p) \right] \tag{4.37a}$$

This means that only the paths

$$\eta_t = -i \sum_p \rho_p \mu_t^p$$
$$\mu_t^p = \bar{G}_{tt_p}^c \tag{4.37b}$$

contribute to the $[\eta_t]$ integration.

Note that the Jacobian $d[\bar{h}_t^c \eta_t]/d[\eta_t] = 1$ by the same arguments as given in Section 3.2. Therefore,

$$\mathscr{M}_{pp'} = \int dt\, dt'\, \mu_t^p \tilde{K}_{tt'} \mu_{t'}^{p'} \tag{4.38}$$

Performing the $\boldsymbol{\rho}$ integration (see footnote 27) in Eq. (4.35), we arrive at

$$\mathscr{R} = \frac{1}{2\pi} |\det \mathscr{M}|^{-1/2} \tag{4.39a}$$

One can show that

$$\det \left[\mathscr{M} - \begin{pmatrix} w_k^{-1} & 0 \\ 0 & w_a^{-1} \end{pmatrix} \right] = 0 \tag{4.40}$$

which is a consequence of the translational zero-frequency mode. Hence,

$$\mathscr{R} = (1/2\pi) |\mathscr{M}_{aa}/w_k + \mathscr{M}_{kk}/w_a - (w_k w_a)^{-1}|^{-1/2} \tag{4.39b}$$

[27] It is always understood that we have to choose integration contours properly in the complex plane in order to ensure convergence.

For the evaluation of the above expression, we return to the symmetric configuration $t_k = -t_0/2$, $t_a = +t_0/2$. For a general orientation, let us also consider the limit $M = 0$. Since we are interested in large separation $t_0 \gg 1$, we note that in order of magnitude

$$\begin{aligned} \mu^k_{-t_0/2}, \mu^a_{t_0/2}, w_k^{-1}, \mathscr{M}_{kk} &\sim 1 \\ \mu^a_{-t_0/2}, w_a^{-1}, \mathscr{M}_{ak} &\sim e^{t_0} \\ \mathscr{M}_{aa} &\sim e^{2t_0} \end{aligned} \tag{4.41}$$

We conclude that only the term $\mathscr{M}_{aa}/w_k$ contributes in Eq. (4.39b).

According to Eq. (4.25) and (4.29), we have

$$w_k = 2\mathscr{B}^k \lambda_2^+ / v_k \tag{4.42a}$$

Also, we recognize that μ_t^a as defined by Eq. (4.37b) is of importance only in the region $t \sim -t_0/2$, i.e., in the region of the kink. There, we have $\bar{h}^k_{(t+t_0/2)} \mu_t^a = 0$; and considering Eqs. (4.7), (4.8), and (4.25), we conclude that

$$\begin{aligned} \mu_t^a &= D\psi^a_{-(t+t_0/2)} \\ D &= \frac{1}{\lambda_2^+} \bar{G}^c_{-t_0/2, t_0/2} = \frac{1}{g} e^{\lambda_2^+ t_0} \end{aligned} \tag{4.42b}$$

Consulting now Eq. (4.10), we find

$$\mathscr{M}_{aa} = -2D^2/\mathscr{B}^k \tag{4.42c}$$

whence we obtain

$$\mathscr{R} = \frac{g}{2\pi} \left(\frac{\lambda_2^+}{v_k} \right)^{1/2} \mathscr{B}^k \cdot e^{-\lambda_2^+ t_0} \tag{4.43}$$

This result allows us to express the prefactor a of Eq. (3.60) by known quantities. Considering the definition (3.55) of $\mathscr{Y}$ and comparing it with Eq. (4.32), we conclude that

$$a = \frac{\omega_0}{\gamma} (v_k v_a)^{1/2} \tag{4.44}$$

Note that the color of the noise enters only indirectly through its influence on the kink velocity v_k. In the white noise limit, this result coincides with Kramers' expression (2.8), since, there, $v_k = v_a = \lambda_2^+$, and $\omega_0 \lambda_2^+/\gamma = a$(Kramers).

In conclusion, for a piecewise parabolic potential (ppp) and for arbitrary color of the noise, we have calculated the decay rate in the form shown in Eq. (1.10) by an asymptotic expansion. We have found that the exponent $b = \mathscr{U}\mathscr{B}^k$, where $\mathscr{U}$ and $\mathscr{B}^k$ are given by Eqs. (3.28) and (4.10), and that the prefactor $a = (\omega_0/\gamma)(v_k v_a)^{1/2}$ depends only on the velocities v_k and v_a of the kink and antikink as given by Eq. (4.16). Recall that the asymptotic expansion is valid under the condition shown in Eq. (1.13); in the present case one may convince oneself by explicit calculation that this condition means a large exponent $b \gg 1$. In Fig. 8, we show a and b as a function of temperature for $M = \omega_0^2/\gamma^2 = 1/9$. Furthermore, the decay rate as found by the present analytical calculation is compared with the results of the numerical simulation in Fig. 6 as a function of $\gamma/\omega_0 = M^{-1/2}$ in the blue noise (BN) limit. Clearly, the agreement is very good if the exponent b is sufficiently large.

4.6. Lorentzian Noise

For the sake of completeness, we demonstrate how to reproduce the results of Luciana and Varga[16] for the Lorentzian noise (LN) source of Eq. (1.15). In terms of the reduced variables of Section 3.4, we have

$$\tilde{K}_\omega^L = 2\theta/(1+\omega^2\tau^2) \qquad \text{(LN)} \tag{4.45}$$

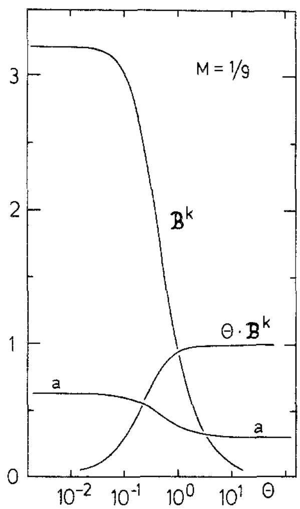

Fig. 8. Temperature dependence of the prefactor a and the exponent $b/\mathscr{U} = \mathscr{B}^k$ for the piecewise parabolic potential (ppp) and for $\gamma = 3\omega_0$ ($M = 1/9$).

For simplicitly, we consider the overdamped limit $M=0$, where Eq. (4.6) reduces to $\psi_\omega^a=2/(1+\omega^2)$. Inserting these relations in Eq. (4.10), we obtain

$$\mathscr{B}^k=\frac{1}{\theta}\frac{(1+\tau)^2}{1+2\tau} \qquad \text{(LN)} \tag{4.46}$$

In order to obtain the prefactor from Eq. (4.44), we calculate the velocities according to Eq. (4.16a) and find that

$$v_a=-1, \qquad v_k=\frac{1}{1+2\tau} \qquad \text{(LN)} \tag{4.47}$$

Returning now to ordinary units, we have for the exponent b and for the prefactor a

$$\begin{aligned} b&=\frac{V_B}{T}\frac{(1+\omega_0^2\tau/\gamma)^2}{1+2\omega_0^2\tau/\gamma} \qquad \text{(LN)} \\ a&=\frac{\omega_0}{\gamma}\frac{1}{(1+2\omega_0^2\tau/\gamma)^{1/2}} \end{aligned} \tag{4.48}$$

This agrees with the results of ref. 16 if the parameters of the ppp are chosen to be the same.

5. THE CUBIC POTENTIAL

5.1. Corrections to the White Noise Limit

For the cubic potential (cp) of Eq. (3.23), we have

$$\begin{aligned} \varphi'(x)&=\tfrac{1}{2}(1-x^2) \\ \varphi''(x)&=-x \\ \varphi'''(x)&=-1 \end{aligned} \tag{5.1}$$

This nonlinearity allows analytical calculations only in the overdamped limit $M=0$ and the white noise (WN) limit. There, we have, according to Eqs. (3.35), (3.36), and (3.38),

$$\begin{aligned} x_t^k&=x_{-t}^a=\tanh t/2 \\ \phi^k&=-\frac{1}{2\theta}\frac{1}{\cosh^2 t/2}; \qquad \phi_t^a=0 \\ \mathscr{B}^k&=\frac{2}{3\theta} \end{aligned} \tag{5.2}$$

The correction to the action can be calculated to lowest order by perturbation theory which is based on the stationarity of the extremal paths. Accordingly, this correction is equal to the expectation value of the perturbation $\Delta\mathscr{B}[x_t, y_t]$ in the action functional taken with respect to the unperturbed paths. In view of the action functional (3.29), we conclude that for $y_t^k = i\phi_t^k$

$$\Delta\mathscr{B}^k = -\frac{1}{2}\int dt\, dt'\, \varphi_t^k \Delta\tilde{K}_{tt'} \phi_{t'}^k \tag{5.3a}$$

where the zeroth-order path is given by Eq. (5.2) and where

$$\Delta\tilde{K}_\omega = \frac{\omega^2}{6\theta} \tag{5.3b}$$

This means that

$$\Delta\mathscr{B}^k = -\frac{1}{12\theta}\int dt (\dot{\phi}_t^k)^2 \tag{5.3c}$$

and consequently

$$\mathscr{B}^k = \frac{2}{3\theta}\left(1 - \frac{1}{60\theta^2}\right) \qquad (\text{WN}, M=0) \tag{5.4}$$

One should compare this result with the corresponding expression (4.11) for the ppp.

5.2. The Prefactor in the White Noise Limit

An exppression for the kink–antikink interaction can be obtained by evaluating the action functional $\mathscr{B}(x_t, y_t = i\phi_t)$ for a combined object (x_t^c, ϕ_t^c) as given by (3.40). Since

$$\begin{aligned} \xi[x_t^c] &= \xi[x_{t-t_k}^k] + \xi[x_{t-t_a}^a] + R_t \\ R_t &= -(1 - x_{t-t_k}^k)(1 - x_{t-t_a}^a) \end{aligned} \tag{5.5}$$

and since $\xi[x_t^a] = 0$, we conclude that in perturbation theory

$$\mathscr{B}_{\text{int}}(t_0) = \mathscr{B}[x_t^c, i\phi_t^c] - \mathscr{B}^k = \int dt\, R_t \phi_t^c \tag{5.6}$$

Inserting the expressions (5.2), we obtain in lowest order

$$\mathscr{B}_{\text{int}}(t_0) = -g\mathscr{B}^k e^{-t_0}, \qquad g = 6 \tag{5.7}$$

In the present case, the ansatz (3.40) works well, since the asymptotics of kink and antikink is of the exponential type. One can also show that corrections to this ansatz contribute to $\mathscr{B}_{\text{int}}$ by a term of second order $\propto e^{-2t_0}$.

In the white noise limit, the Gaussian fluctuations about the combined object can be obtained conveniently from Eq. (3.56). Since the contributions of kink and antikink are essentially the same (by some kind of time-reversal symmetry), we have[28]

$$\frac{1}{\beta_k \beta_a}\left(\frac{\det'' \mathscr{H}^c}{\det \mathscr{H}^-}\right)^{1/2} = \frac{1}{\beta_a^2}\frac{\det' \mathscr{H}^a}{\det \mathscr{H}^-}\, e^{t_0} \tag{5.8}$$

This can be understood as follows. First, we note that

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BC) \tag{5.9}$$

if any of the submatrices is proportional to the unit matrix. Therefore,

$$\begin{aligned} \det' \mathscr{H}^p &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \det \begin{pmatrix} -\phi_t^p + \varepsilon & \bar{h}_t^p \\ h_t^p & 2\theta + \varepsilon \end{pmatrix} \\ &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \det\left[-\bar{h}_t^p h_t^p - 2\theta\phi_t^p + 2\theta\varepsilon\left(1 - \frac{1}{2\theta}\phi_t^p\right)\right] \end{aligned} \tag{5.10}$$

where $\tilde{K}_{tt'} = 2\theta\delta(t - t')$ has been inserted in $\mathscr{H}^p$ of Eq. (3.48). Taking into account the properties of the extremal path [see Eq. (5.2)], we find that

$$\bar{h}_t^p h_t^p + 2\theta\phi_t^p = \bar{h}_t^a h_t^a; \qquad p = k, a \tag{5.11}$$

Using the fact that $\bar{h}_t^a h_t^a \dot{x}_t^a = 0$, we determine the ε contribution to the determinant by perturbation theory. Thus,[29]

$$\begin{aligned} \det' \mathscr{H}^a &= 2\theta \det' \bar{h}^a h^a \\ \det' \mathscr{H}^k &= 2\theta \frac{\beta_k}{\beta_a} \det' \bar{h}^a h^a \end{aligned} \tag{5.12}$$

which proves Eq. (5.8). At this point, we observe that

$$\begin{aligned} \bar{h}_t^a h_t^a &= [-\partial_t^2 + W''(x_t^a)] \\ W(x) &= \frac{1}{8}(1 - x^2)^2 \end{aligned} \tag{5.13}$$

[28] The additional factor is 1 for $M = 0$. In addition, we have to supply a factor e^{t_0} which compensates the discontinuity of $M \to 0$ and $M = 0$.

[29] Note that for the normalization (3.39), we have to compare expressions of the type $4\int dx \sinh^2 x/\cosh^6 x = \int dx(1/\cosh^6 x)$.

and that the fluctuations about the instantons of the double-well potential $W(x)$ have been done "doubly well" in ref. 24. Accordingly,

$$\frac{\det' \bar{h}^a h^a}{\det \bar{h}^- h^-} = \frac{1}{12} \tag{5.14}$$

and we obtain from Eqs. (3.38), (3.39), (3.55), (5.8), and (5.14)

$$\frac{1}{\beta_k \beta_a} \mathscr{Y} = \frac{1}{4} \theta \tag{5.15}$$

Using Eqs. (3.60), (5.4), and (5.7), we find for the prefactor

$$a = \frac{\omega_0}{\gamma} \tag{5.16}$$

which agrees with Eq. (2.8) in the overdamped limit $M = 0$.

5.3. Numerical Calculations of the Action

The equations (3.33) for extremal paths can be combined in the form

$$[M\partial_t^2 - \partial_t + \varphi''(x_t^k)] \int dt' \, \tilde{N}_{tt'} [M\ddot{x}_{t'}^k + \dot{x}_{t'}^k + \varphi'(x_{t'}^k)] = 0 \tag{5.17}$$

where we have noted explicitly our interest in the kink.

In order to cast Eq. (5.17) in a manageable form, we have to assume strong damping, $M = 0$. For convenience, we introduce $x_t^k = -2\dot{w}_t / w_t$ and obtain

$$\begin{aligned} \dot{x}_t^k + \varphi'(x_t^k) &= -2 \frac{\ddot{w}_t}{w_t} + \frac{1}{2} \\ -\partial_t + \varphi''(x_t^k) &= -w_t^2 \frac{\partial}{\partial t} \frac{1}{w_t^2} \end{aligned} \tag{5.18}$$

where we have made use of Eq. (5.1). Then, Eq. (5.17) can be integrated once and we find

$$-\ddot{w}_t + \frac{1}{4} w_t = \frac{1}{2} D w_t \int dt' \, \tilde{K}_{tt'} w_{t'}^2 \qquad (M = 0) \tag{5.19}$$

where the integration constant D can be chosen at convenience. Note also that the action of Eq. (3.30) can be written as

$$\mathscr{A}^k = \mathscr{B}^k = \frac{1}{2} D^2 \int dt \, dt' \, w_t^2 \tilde{K}_{tt'} w_{t'}^2 \tag{5.20}$$

For orientation, we consider the white noise limit where $\tilde{K}_{tt'} = 2\theta\delta(t-t')$. Choosing $D = 1/2\theta$, we obtain

$$\ddot{w}_t - \tfrac{1}{4} w_t + \tfrac{1}{2} w_t^3 = 0 \qquad (\text{WN}, M=0) \tag{5.21}$$

The solution to this equation is $w_t = (\cosh t/2)^{-1}$; and we may confirm relation (5.2).

For a numerical calculation, we have written Eq. (5.19) in terms of Fourier transforms. The iteration procedure was similar to the one proposed in ref. 30. We have also made use of the possibility to choose the integration constant to our convenience. The error of our results (e.g., discretization) is estimated to be less than 1%.

In addition, the kink action can also be calculated on the basis of a variational principle. There, we define the functional

$$\hat{\Sigma}[w_t] = 2D \int dt \left(\dot{w}_t^2 + \frac{1}{4} w_t^2 \right) - \frac{1}{2} D^2 \int dt\, dt'\, w_t^2 \tilde{K}_{tt'} w_{t'}^2 \tag{5.22}$$

and then we observe that $\delta\hat{\Sigma}/\delta w_t = 0$ is equivalent to Eq. (5.19). In addition, choosing D to take the value which maximizes $\hat{\Sigma}$, we obtain

$$\Sigma[w_t] = 2 \frac{[\int dt(\dot{w}_t^2 + w_t^2/4)]^2}{\int dt\, dt'\, w_t^2 \tilde{K}_{tt'} w_{t'}^2} \tag{5.23}$$

Obviously, the expression above is invariant under $w \to \alpha w$, where α is a constant. For $\theta = 0$ (BN), the denominator in Eq. (5.23) is invariant with respect to a change in time scale; therefore, the optimal scale can be chosen easily. In this final form, the dependence on the form of w_t is weak. For instance, if we take the form $w_t = (\cosh t/2)^{-1}$, we obtain a value for $\mathscr{B}^k$ which is only 3% larger than the one obtained by iteration.

The numerical results are summarized in Table I. For blue noise, we have $\frac{3}{2}\mathscr{B}^k = 4.18$, which leads, for large damping $\gamma \gg \omega_0$ ($M \to 0$), to the expression for $b = \mathscr{U}\mathscr{B}^k$ as given in Eq. (2.9).

In Fig. 5 we have compared the logarithmic decay rate $\sigma = \ln(\omega_0/2\pi\Gamma)$ as obtained from the numerical simulations (cp) with the present calculation. As we have not succeeded in calculating the prefactor a for blue noise (BN), we have taken $a = 1$. The agreement is reasonable and we may attribute the difference to the prefactor.

Table I. Temperature Dependence of the Exponent $3b/2\mathscr{U} = 3\mathscr{B}^k/2$ for the Cubic Potential and for $M = 0$

θ	10^{-4}	10^{-3}	10^{-2}	10^{-1}	10^{0}	10^{1}
$\frac{3}{2}\mathscr{B}^k$	4.18	4.17	4.16	3.93	0.984	0.0996

6. QUANTUM THEORY OF THERMAL ACTIVATION

In this section we discuss some of the consequences which follow from the full quantum mechanical action $\mathscr{B}_{\rm QM}[x_t, y_t]$ shown in Eq. (3.17). In terms of the reduced variables introduced in Section 3.4, we have $\mathscr{B}_{\rm QM}[x_t, y_t] \to \mathscr{U}\mathscr{B}^*_{\rm QM}[x^*_t, y^*_t]$, where for the cubic potential (cp) (omitting henceforth asterisks again)

$$\begin{aligned} \mathscr{B}_{\rm QM}[x_t, y_t] &= \mathscr{B}[x_t, y_t] + \Delta\mathscr{B}_{\rm QM}[x_t, y_t] \\ \Delta\mathscr{B}_{\rm QM}[x_t, y_t = i\phi_t] &= -i\frac{1}{24}\int dt\, y_t^3 = -\frac{1}{24}\int dt\, \phi_t^3 \end{aligned} \tag{6.1}$$

and where $\mathscr{B}[x_t, y_t]$ is given by Eq. (3.16) with $\xi_t = M\ddot{x}_t + \dot{x}_t + \varphi'(x_t)$.

Again, we will evaluate path integrals of the type (3.15b) with the integrand $Q_{\rm QM} = \exp(-\mathscr{U}\mathscr{B}_{\rm QM})$ in the extremal path approximation. We have already anticipated the fact that for such paths, y_t is purely imaginary.[30] Explicitly, the equation of motion for the extremal path is now

$$\begin{aligned} (M\partial_t^2 - \partial_t - x_t)\,\phi_t &= 0 \\ \left[M\ddot{x}_t + \dot{x}_t + \frac{1}{2}(1 - x_t^2)\right] + \frac{1}{8}\phi_t^2 + \int dt'\, \tilde{K}_{tt'}\phi_{t'} &= 0 \end{aligned} \tag{6.2}$$

We recognize immediately that there is an extremal path $(x_t^a, \phi_t^a = 0)$ representing an antikink which is the same as in the quasiclassical approximation. We may also convince ourselves that for $\theta \gg 1$, there is a kinklike path of the type (5.2) except for corrections of relative order θ^{-2}.

In fact, it is even possible to find the explicit form of a kink in the strong damping limit $M = 0$ provided that $\theta > \theta_B$. Let us look for a solution of the form

$$\begin{aligned} \phi_t^k = cf_t; \qquad x_t^k &= -\dot{f}_t/f_t \\ f_t &= (\cosh t + \cos\tau)^{-1} \end{aligned} \tag{6.3}$$

First, we observe that

$$\begin{aligned} f_t^2 &= \frac{1}{\sin\tau}\frac{\partial}{\partial\tau} f_t \\ f_\omega &= \frac{2\pi}{\sin\tau}\frac{\sinh\omega\tau}{\sinh\omega\pi} \\ (f_t^2)_\omega &= (\omega\coth\omega\tau - \coth\tau)\, f_\omega/\sin\tau \end{aligned} \tag{6.4}$$

[30] Indeed, this fact forces us to discard the piecewise parabolic potential (ppp) and to work with the cubic potential (cp) in spite of the difficulty of obtaining analytical solutions for the extremal paths.

Inserting the above relations in Eq. (6.2), we find that for $M=0$, the equation of motion is satisfied if $\tau = 1/2\theta$ and $c = -2\sin\tau$. Hence

$$\phi_t^k = -2\,\frac{\sin\tau}{\cosh t + \cos\tau}$$
$$x_t^k = \frac{\sinh t}{\cosh t + \cos\tau} \tag{6.5}$$
$$\tau = 1/2\theta; \qquad \theta > \theta_B = 1/2\pi$$

For $\theta \gg 1$, this form of a quantum kink agrees in leading order with the quasiclassical approximation (5.2). Next we insert Eq. (6.5) in Eq. (6.1), whence we obtain the simple result for the action

$$\mathscr{B}^k_{\mathrm{QM}} = \frac{4}{3}\,\tau = \frac{2}{3\theta} \tag{6.6}$$

Compare this expression with the high-temperature result (5.4) of the QCL where a correction term of order θ^{-3} appears. In order to understand this difference, we calculate the difference $\Delta\mathscr{B}_{\mathrm{QM}}$ by perturbation theory. Inserting the unperturbed paths of Eq. (5.2), we obtain

$$\Delta\mathscr{B}^k_{\mathrm{QM}} = -\frac{1}{24}\int dt(\phi_t^k)^3 = \frac{1}{90}\frac{1}{\theta^3} \tag{6.7}$$

which is exactly the difference between Eq. (6.6) and (5.4).

A similar compensation occurs for all types of metastable potentials. Consider, for instance, the ppp, where in leading order

$$\Delta\mathscr{B}_{\mathrm{QM}}[x_t, y_t] = -\frac{i}{12}\int dt\,\delta(x_t)\,y_t^3$$
$$\Delta\mathscr{B}^k_{\mathrm{QM}} = \frac{1}{12}\frac{(\lambda_2^+)^2}{\theta^3} \tag{6.8}$$

which compensates the second-order term of the QCL expansion (4.11).

We conclude: The decay rate of the QCL agrees with the result of the classical Langevin equation to leading order in the white noise limit $\theta \to \infty$. However, the next order corrections to the exponent of the decay rate are an artefact of the quasiclassical approximation (1.7).

Concerning the prefactor, we may form a combined object (x_t^c, ϕ_t^c) as shown in Eq. (3.40) and calculate the kink–antikink interaction in the same way as done in Section 5.2. Then we obtain a form as shown in Eq. (5.7), where, however,

$$g_{\text{QM}} = 6\,\frac{\sin\tau}{\tau} \tag{6.9}$$

Proceeding further, we decompose the fluctuation determinant of the combined object as in Section 5.2 and observe that the antikink contribution is the same. Eventually, we arrive at

$$a = \frac{\omega_0}{\gamma}\,\frac{\beta^k_{\text{QM}}}{g_{\text{QM}}}\,3\theta^{1/2}\left(\frac{\det'\mathscr{H}^k_{\text{QM}}}{\det\mathscr{H}^-}\right)^{1/2} \tag{6.10}$$

where the kink normalization is given by

$$(\beta^k_{\text{QM}})^2 = \frac{5}{\sin^2\tau}\left(1-\frac{\tau\cos\tau}{\sin\tau}\right)-1 \tag{6.11}$$

Furthermore, the fluctuation operator is given by

$$\mathscr{H}^p_{\text{QM}} = \mathscr{H}^p_{tt'} + \frac{1}{4}\,\phi^p_t\,\delta(t-t')\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \tag{6.12}$$

with the quasiclassical contribution as shown in Eq. (3.48).

At this point, it is appropriate to recall some results of the standard quantum theory[31] as applied to thermal activation.[32] There, one calculates the imaginary part of the free energy and one agrees that, for $T > T_B$, there is only a contribution from the trivial orbit[31] where the particle rests on the top of the barrier. This leads immediately to the same exponent $\mathscr{B}^k = 2/3\theta$, $b = V_B/kT$ as here. However, the Gaussian fluctuations about the trivial orbit can easily be evaluated. This leads to the prefactor

$$a_{\text{tr}} = \frac{\omega_0}{\gamma}\left|\prod_{n=-\infty}^{+\infty}\frac{\nu_n^2+\gamma\,|\nu_n|+\omega_0^2}{\nu_n^2+\gamma\,|\nu_n|-\omega_0^2}\right|^{1/2} \tag{6.13}$$

where $\nu_n = 2\pi kTn/\hbar$.

We have not been able to evaluate the fluctuation determinant in Eq. (6.10); therefore, a comparison with the expression (6.13) cannot be made.

We have also been unable to solve the equation of motion (6.2) for $T < T_B$. In one attempt, we tried to find a kink solution numerically by

[31] Since the free energy is calculated in imaginary (Euclidean) time, there is no obvious connection between the orbits there and in the present real-time theory.

methods similar to the one discussed in Section 5.3. For a control, we substituted $\Delta\mathscr{B}_{QM} \to \varepsilon\Delta\mathscr{B}_{QM}$, where ε was slowly increased in the course of iterations. However, an instability in the iteration at $\varepsilon \sim 0.9$ prevented us from forming definite conclusions.

7. QUANTUM NOISE

In their experiment Koch *et al.*[5] were able to observe quantum noise of an Ohmic resistor at high frequencies $\omega \gg kT/\hbar \sim 10^{11}$ Hz. According to their interpretation, the Josephson junction should be considered as a nonlinear device which mixes the high-frequency noise down by the Josephson frequency $2eU/\hbar$ to the low measurement frequency $\sim 10^5$ Hz.

We recall that a Josephson junction shunted by an Ohmic resistor (resistively shunted junction, RSJ) corresponds to a Brownian quantum particle where the phase difference (in the range $-\infty < \varphi_t < \infty$) has to be identified with the position x_t. Further characteristic data are (see, e.g., ref. 33)

$$\begin{aligned} m &= (\hbar/2e)^2\, C; \qquad m\gamma = (\hbar/2e)^2/R \\ V(x) &= -E_J \cos x - Fx \\ E_J &= (\hbar/2e)\, I_J; \qquad F = (\hbar/2e)\, I_X \end{aligned} \tag{7.1}$$

In the relation above, C is the capacitance and I_J the critical current of the junction. Furthermore, R is the shunt resistor and I_X an externally applied bias current. Note that the Josephson frequency (at $I_X = 0$) is given by

$$\omega_0^2 = E_J/m = 2eI_J C/\hbar \tag{7.2}$$

In the scheme of reduced variables presented in Section 3.4, we leave the (presently) dimensionless coordinate $x_t = \varphi_t$ unchanged, but scale time and frequency as shown there. Furthermore,

$$\mathscr{U} = \frac{m\gamma}{\hbar} = \frac{\hbar}{4e^2 R} = \frac{6.5k\Omega}{R} \tag{7.3}$$

and we obtain from Eqs. (3.17) and (3.18) (asterisks omitted)

$$\begin{aligned} \mathscr{B}_J[x_t, y_t] = i \int dt \left\{ y_t[M\ddot{x}_t + \dot{x}_t - F] + 2 \sin\frac{y_t}{2} \sin x_t \right\} \\ + \frac{1}{2} \int dt\, dt'\, y_t \tilde{K}_{tt'} y_{t'} \end{aligned} \tag{7.4}$$

In order to have a simple reference to the model, we have replaced $(\mathscr{B}_J)_{QM}$ by $\mathscr{B}_J$.

By the Josephson relation, $2eU = \hbar\dot{\varphi}$; therefore, the voltage is proportional to the velocity $\dot{x}_t$. Hence, the measured voltage noise has a power spectrum which is equal to the velocity–velocity correlation

$$S_{tt'} = \langle \dot{x}_t \dot{x}_{t'} \rangle - \langle \dot{x}_t \rangle \langle \dot{x}_{t'} \rangle \tag{7.5}$$

In a standard way, this correlation function can be computed by functional differentiation of the generating functional (see footnote 17)

$$Z[f_t] = \int d[x_t, y_t] \exp - \left\{ \mathscr{B}_J[x_t, y_t] + i \int dt\, f_t \dot{x}_t \right\}$$

Thus,

$$S_{tt'} = - \left. \frac{\delta^2 \ln Z[f_t]}{\delta f_t \delta f_{t'}} \right|_{[f_t = 0]} \tag{7.6}$$

Let us calculate Z asymptotically for $\mathscr{U}$ very large. In a first step, we have to determine the extremal paths for $[f_t = 0]$. Putting $y_t = i\phi_t$, these paths are determined by

$$\begin{gathered} (M\partial_t^2 - \partial_t)\, \phi_t + 2 \sinh \frac{\phi_t}{2} \cos x_t = 0 \\ (M\ddot{x}_t + \dot{x}_t - F) + \cosh \frac{\phi_t}{2} \sin x_t + \int dt'\, \tilde{K}_{tt'} \phi_{t'} = 0 \end{gathered} \tag{7.7}$$

For $F > F_c$, there is a nontrivial path which is a solution of the deterministic equation of motion representing a free running ($p = r$) Josephson junction, where $\phi_t^r = 0$ and $\langle \dot{x}_t^r \rangle \neq 0$. Specifically, we have

$$M\dot{x}_t^r + \dot{x}_t^r - F + \sin x_t^r = 0 \tag{7.8}$$

Expanding $\mathscr{B}_J$ quadratically about this path, we have, since $\mathscr{B}_J^r = 0$,

$$\mathscr{B}_J = \mathscr{L}^r[\zeta_t, \eta_t] \tag{7.9}$$

where $\mathscr{L}^r$ is of the form (3.47) with $\mathscr{H}^r$ as given (since $\phi_t^r = 0$) in (3.48). Note also that Eq. (3.49) remains meaningful,

$$h^r = M\partial_t^2 + \partial_t + \cos x_t^r \tag{7.10}$$

as well as the Green's function $G_{tt'}^r$, which is found as a solution of

Eq. (3.50), and which is bounded in the present case. In addition, we also need the inverse of $\mathscr{H}^r$, which can be written as follows:

$$\mathscr{G}^r_{tt'} = \begin{pmatrix} 0 & G^r_{tt'} \\ \bar{G}^r_{tt'} & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \int dt_1\, dt'_1\, G_{tt_1} \tilde{K}_{t_1 t'_1} \bar{G}_{t'_1 t'} \tag{7.11}$$

Therefore, we conclude that within the present approximation, the generating functional is given by

$$\begin{aligned} Z[f_t] &= \int d[\zeta_t, \eta_t] \exp - \left\{ \mathscr{L}^r[\zeta_t, \eta_t] + i \int dt [\dot{x}^r_t + \dot{\zeta}_t] f_t \right\} \\ &= \exp - \left\{ \frac{1}{2} \int dt\, dt'\, \dot{f}_t (\mathscr{G}^r_{tt'})_{11} \dot{f}_{t'} - i \int dt\, \dot{x}^r_t f_t \right\} \end{aligned} \tag{7.12}$$

where $(\mathscr{G}^r_{tt'})_{\alpha\alpha'}$ means the (α, α') matrix element of $\mathscr{G}^r_{tt'}$. Clearly, we have

$$S_{tt'} = \frac{\partial^2}{\partial t\, \partial t'} (\mathscr{G}^r_{tt'})_{11} = \frac{\partial^2}{\partial t\, \partial t'} \int dt_1\, dt'_1\, G_{tt_1} \tilde{K}_{t_1 t'_1} \bar{G}_{t'_1 t'} \tag{7.13}$$

It is not difficult to convince oneself that the quasiclassical approximation (1.7) which replaces the term $2 \sin \frac{1}{2} y_t \sin x_t$ in Eq. (7.4) by $y_t \sin x_t$ does not change the result (7.13). Therefore, we conclude that within the accuracy of the semiclassical approximation, *the QCL does lead to the correct result for quantum noise.* In fact, the result (7.13) can be obtained from Eq. (1.1) for a potential $V(x) = -E_J \cos x - Fx$ to lowest order in the noise source, as demonstrated explicitly in ref. 4 for a strongly damped Josephson junction ($M = 0$).

As a side remark, we mention that a Gaussian noise in the bias current with correlation $\langle \delta I_X(t)\, \delta I_X(t') \rangle = (2e/\hbar)^2 \tilde{L}_{tt'}$ contributes to the observed voltage noise $S_{tt'}$ such that $\tilde{K}_{tt'}$ in Eq. (7.13) is replaced by $\tilde{K}_{tt'} + \tilde{L}_{tt'}$.

8. CONCLUSIONS

We have been motivated to undertake the investigations presented in this paper by questions concerning the quality of the quasiclassical approximation (1.7) which leads to the quasiclassical Langevin (QCL) equation (1.1). In ref. 6, it has been argued that for sufficiently large damping ($\gamma \gg \hbar/mx_B^2$) one may expect such an approximation to be reasonable.

Concerning the decay of a metastable state, which we have studied here most extensively, this expectation is justified only within a very restricted interpretation. With increasing damping, the crossover temperature T_B decreases, so that for a given temperature, the Brownian particle behaves more and more classically. On the other hand, we cannot help but admit our disappointment that the QCL produces corrections to the high-temperature limit which are spurious. At very low temperature, the QCL leads for the cp to an exponent b of the decay rate (1.10) which is of the same functional form as the one obtained from the instanton technique but which is smaller by about a factor 1/2.[32]

On the other hand, we have shown by an explicit calculation with a Josephson junction as a model that the QCL reproduces correctly the influence of quantum noise on a predominantly deterministic motion of the Brownian particle.

From a formal point of view, the Langevin equation is an equation of motion for the position $x(t)$ of the Brownian particle driven by a stochastic force $\xi(t)$. We have a strong increase of noise power with frequency (blue noise), which may be considered to be a consequence of zero-point fluctuations. In this case, it is necessary to retain the acceleration term $m\, d^2x(t)/dt^2$ in the Langevin equation. It is a remarkable feature that this acceleration term changes the Jacobian of the mapping $\xi(t) \to x(t)$ in a drastic way independent of the smoothness or ruggedness of the stochastic process $\xi(t)$. Furthermore, this paper opens an interesting perspective, since it provides a bridge between the path integral description of Langevin processes with colored noise[(20)] and the Keldysh technique[(10)] for a real-time presentation of quantum processes. In particular, we propose to investigate more closely the consequences of this ansatz for quantum decay problems.

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[32] We acknowledge gratefully that Dr. August Ludviksson from our Institute has carried through calculations concerning the quantum decay for the ppp. His result (obtained by numerical calculations and for large damping) is $b = 9.5\mathscr{U}$. On the other hand, we obtain from the QCL $b = \pi\mathscr{U}$ in the same limit. Hence, the ratio is $\pi/9.5 \sim 1/3$ in this case.

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